

The Witness and the Rose, a CT-HoTT Fable

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This is *The Little Prince* retold as a gentle proof: a story where pictures carry laws, promises carry proofs, and love is a path you can draw. You can read it straight—sand, stars, fox, rose—or, if you like, glance at the parenthetical CT/HoTT “code” that accompanies each scene. Those small asides name the shapes behind the magic: a triangle-in-square for daily care (face = law, diagonal = choice, filler = witness), a resource square for the planet’s baobabs, a cube whose interior coherence binds time and capacity, an adjunction for taming the fox, a pushout for the well in the desert, a collar movie for saying goodbye. None of the math is required; it’s there as a quiet scaffold, a way to notice that what makes something *yours* is a witness you build over time. If you’ve ever felt that rituals make meaning, that responsibility is a theorem of love, or that laughter can transport stars into joy, you already speak this language. Turn the page; bring your pencil. We’ll draw the faces first—and keep our witnesses.

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1) Sand, Stars, and a Box

The engine gave out the way a heart gives out—silently, with one last cough you pretend not to hear. I landed the plane on a page of sand so blank it made my instruments feel like lies. Night fell like a lid. The stars came on, uncountable, and my toolbox sounded louder than the wind.

At dawn a voice as light as a bird's foot on water said, "Please... draw me a sheep."

I turned. A child stood there as if the desert itself had been thinking and decided to speak. Hair the color of wheat I hadn't seen for a thousand miles. Eyes that contained no doubt, only a request.

"I don't know how to draw sheep," I said, because grown-ups always begin by honoring their limits.

He waited, not politely, just completely.

I tried a sensible sheep: horns wrong. Another: too old. A third: too sickly. The sun climbed. My canteen warmed. The child kept rejecting each animal with a grave little shake of the head, as if I had misread a theorem whose statement was obvious.

(Between **Need** and **Means** there was no witness: a face sketched but unfilled. In our terms: no 2-cell yet. Parenthetical note: (missing filler between Need and Means).)

"Make me a sheep," he said again, as if repetition might conjure a bridge.

A memory pushed up through the sand: the drawing I had made at six—a boa constrictor digesting an elephant—that adults insisted was a hat. I began to understand. He wasn't asking for fur and legs. He was asking for *satisfaction*, for a certain *mapping* from a quiet criterion to a possible world.

So I drew a box. Three quick lines and a little handle. "The sheep you want is inside."

He took the page as if receiving a key. His face cleared; something clicked into place, not on the paper but in the air between us.

(CT/HoTT aside: the *box* stands for a universal solution. Instead of giving one specimen $t : T$ that might fail the spec, we offer an initial object among all drawings that satisfy it. Code:

$\text{Spec} := \sum s:\text{Sheep}. \text{Healthy}(s) \wedge \text{Small}(s)$

$U : \text{Sheep} \rightarrow \text{Drawings}$

$\square : \text{Initial in } (U \downarrow \text{Spec})$

Acceptance = “for every $s:\text{Spec}$, there exists a unique arrow $s \rightarrow \square$.”)

“You see,” he said, tracing the corner of the box, “this one is exactly right. He can eat the baobabs if he tries, but he won’t become too big.”

“You have baobabs?” I asked, thinking of trees so wide three men couldn’t hug them.

He nodded, solemn. “On my planet you must pull them up every morning. It is very easy—unless you forget.”

We sat cross-legged in the resource of shade my broken wing made. He studied the box as though the sheep might bleat. I studied him in the same way.

When he smiled at last, it was for both of us. “Thank you,” he said. “Now we may continue.”

“Continue what?”

“The proof,” he said, as if it were ordinary to speak of proofs on a Tuesday in a desert. “You drew me a sheep that satisfies the face. We can fill the rest later.”

(Aside: He had accepted the **universal property**, not a fragile instance. The face is a *law*; the filler is a *witness*. We had the first: a triangle with a diagonal, waiting for its 2-cell. Code hint: (face declared; filler pending).)

I wanted water, and a miracle. The miracle had arrived wearing a scarf.

We inventoried the visible world: one plane, one boy, one sun, one shadow, one box. He held each item like a lemma, then tucked it back into silence. His questions arrived like well-aimed pebbles: Where did I fall from? Was my planet

large? Did I grow roses? Did I know how to fix things? Each answer he accepted with the courtesy of a student who is certain of the final result.

“Are you afraid?” I asked, when the wind paused long enough for the question to land.

“Only of forgetting,” he said. “If you forget to pull a small baobab, it becomes a large one, and then it is too late.”

I thought of duties I had let harden into habits, and habits I had let erode into excuses.

He pressed a finger to the box. “He will be all right here tonight?”

“Yes,” I said, surprised at my own certainty. “There are holes in the box. He can breathe.”

He nodded, satisfied by the existence of an invisible but necessary morphism.

(Aside: He was checking for a *map* from “sheep” to “air” that keeps “life” invariant. In tiny code: Air: Resource, Breathe: Sheep $\rightarrow$ Air, witness that Breathe(s) $\leq$ Available(Air) holds. ($p : D \circ F \leq A$)—a resource face in miniature.)

We watched the sun make its long arc. The heat was a statement. The silence, its proof. Somewhere a fly wanted what we had and could not find it: a patch of shade, the feeling that two points on a map had decided to be neighbors.

As the shadow thinned, I found my hand on the toolbox again, though I no longer believed in wrenches alone. “If I fix the plane,” I said, “will you still want your sheep?”

He looked injured by the thought of alternatives. “Of course. A plane is to fly. A sheep is to keep promises.”

“And what are promises for?”

“To become responsible for what we love,” he said, as if reading from a book I had not been given.

He curled on the sand then, one knee folded as if listening to music the desert stored under its skin. Sleep arranged his face into an earlier version of itself. The box lay between us like a solved problem, but I knew better: the answer had only pulled a larger question into view.

(Aside: We now have a *partial diagram*: Need—(witness?)—Means. A candidate diagonal F has been chosen (the box drawing mediates the request), the law (face) is specified by the child's spec, and a future $\alpha : C \circ F \Rightarrow G$ will need to be proven: "chosen tasks produce the right kind of time/attention." The story has begun to draw itself as:

S (structure yet unnamed), T (tasks not yet listed), Time (a poset we will call ritual).)

By the last blue of evening, the stars had come back in their old clothes, pretending they had always been there. I leaned against the fuselage and listened to the clicking of hot metal becoming cool metal. The child slept, but his request did not. It had settled into the camp like a compass.

Tomorrow, I thought, we will name the corners. We will draw the square. We will find the diagonal. And if we are very careful with our faces, perhaps the desert will allow a filler.

2) Triangle in a Square

We drew in the sand because paper felt too small for what he meant. With a stick he marked three points and named them as if introducing friends: **Structure**, **Tasks**, **Time**.

"My planet," he said, tapping the first corner, "is not very big. Three volcanoes like cups on a saucer. One is extinct, but that is no reason not to sweep it. And there is my rose. She is very proud, and very delicate. The wind makes her cough." He drew a tiny dome over an imaginary flower and looked relieved, as if the mere gesture could help across this distance.

(Parenthetical code)

S : Type -- Structure

S := Planet $\oplus$ {rose : Rose, glass : Cloche, wind : Screen, volcanoes : 3}

At the second corner he made a stack of dots. “These are what I do each morning: water the rose, clean her glass, set up the wind screen, pull up the baobabs, sweep the volcanoes.” He smiled. “It is not terrible. It is only *exact*.”

(Parenthetical code)

T : Type -- Tasks

T := List(Task)

Task := Water | PolishGlass | SetScreen | Uproot(Baobab) | Sweep(Volcano)

At the third corner he drew a sun and then a small clock. “Time is what makes everything real,” he said. “If you come at four o’clock, I will begin to be happy at three.”

(Parenthetical code)

Time : Poset -- attention/ritual units

(Time, +, 0, <=) -- monoidal poset

He connected **Structure** to **Tasks** with a diagonal and wrote, in careful print, a single letter: **F**. “This is how I choose what to do from what I have,” he explained. “If there is a draft, I set up the screen. If there is sand on the volcano, I sweep.”

(Parenthetical code)

F : S -> T -- choose routine from situation

Then he traced the top edge from **Tasks** to **Time** and labeled it **C**. “Watering takes time. Sweeping takes time. Even looking takes time. This tells how much.”

(Parenthetical code)

C : T -> Time -- cost/schedule

Finally, he tapped the left edge from **Structure** to **Time** and wrote **G**. “This is what my world asks from me each day. If I do this much, she does not cough, and the volcanoes do not grumble, and the baobabs do not split the planet. It is the measure my love becomes.”

(Parenthetical code)

$G : S \rightarrow \text{Time}$ -- demanded ritual/attention

He stood back: three corners, two sides, one diagonal. The square waited, like a face that knows a joke but won't tell it. "Now we need a *certificate*," he said, eyes bright. "We must show that the time my tasks take is enough for what my world asks."

He knelt and filled the center with hatch marks until the sand looked like woven cloth. "This is the witness. If I choose tasks by **F**, and count them by **C**, they will meet the time **G** says is right for my rose."

(Parenthetical code)

$\alpha : C \circ F \Rightarrow G$ -- 2-cell/witness

Default orientation (lax): $C(F(s)) \leq G(s)$

Strict (rare, but possible): $C(F(s)) = G(s)$

He looked at me to see if I was keeping up. I drew in the blank corner and wrote **P**.

"What is *P*?" he asked.

"The process object," I said, surprising myself. "The corner where your structure and your tasks meet with a proof between them."

He clapped, delighted. "Yes! The place where it is *ours*."

(Parenthetical code)

$P := \sum s:S. \sum t:T. \text{Id_Time}(C(t), G(s))$ (*strict homotopy pullback*)

or

$P := (G \downarrow C) = \sum s:S. \sum t:T. (C(t) \leq G(s))$ (*lax/comma*)

"And the diagonal?" he asked.

"It factors through *P*," I said, and we drew a small arrow that dropped from **Structure** into the woven middle before rising to **Tasks**.

(Parenthetical code)

$\phi : S \rightarrow P$ with $\phi(s) := (s, F(s), \alpha_s)$

Projections: $p_1:P \rightarrow S$, $p_2:P \rightarrow T$ satisfy $p_1 \circ \phi = \text{id}_S$, $p_2 \circ \phi = F$.

He nodded as if we had affixed a plaque. “When I water my rose, I do not pour minutes on her leaves,” he said softly. “But time appears anyway. The dome becomes clear. The wind screen stands. The planet feels quiet. That is the proof.”

We rehearsed a morning to see the geometry breathe. He acted it out, hands quick, voice steady. “Wake. Look at sky. If sand is on the volcanoes, sweep. (He mimed gentle circles.) If wind, set the screen. If the glass is dusty, polish. Water, a little. And the baobabs—always the baobabs.” His face shadowed. “If you forget even once, they remember for you.”

I watched his sequence braid itself from the diagonal into the top edge: choices becoming minutes. The left edge—structure to time—felt like a heartbeat under the whole figure.

(Parenthetical code)

Example instance:

$s_0 : S$ (today’s structure)

$F(s_0) = [\text{Uproot, Sweep, SetScreen, Water, PolishGlass}]$

$C(F(s_0)) = 35 \text{ min}$

$G(s_0) = 40 \text{ min}$

$\alpha_{\{s_0\}} : 35 \leq 40$ (*witness inhabits Hom_Time*)

“What if you are sick?” I asked. “What if the wind rises too high for the screen?”

“Then the face is unfilled,” he said simply. “Or filled the other way.” He drew a tiny crack across the hatched center. “It happens. But if I have the diagram, I can see *where* to fix it.”

He redrew the diagonal more carefully this time, as if accuracy of F could make up for weather. “Sometimes I am tempted to add unnecessary tasks to look busy,” he confessed. “Then $C(F(s))$ becomes too large. Or sometimes I pretend the rose will be well without water, and then $G(s)$ becomes large while $C(F(s))$ stays small.” He shook his head. “The diagram does not lie. It shows me the gap.”

(Parenthetical code)

Slack metric: $\text{slack}(s) := G(s) - C(F(s))$ (expect ≥ 0 in our orientation).

If $\text{slack}(s) < 0$, witness α_s cannot be built; diagonal fails to lift into P .

He smoothed the sand and drew the square again, smaller, like a card one might carry. “This is only the face for time,” he said. “There is another face for resources—the baobabs belong there. But we will keep our promise to the square first.”

We sat while the sun tugged shadows across the diagram like threads. “You see,” he said, “grown-ups ask for numbers first. But a number is a stranger until you show me its face and tell me who vouches for it.”

“I have been a grown-up all my life,” I admitted.

He considered this with kind cruelty. “Then you will enjoy being a child for a few pages.”

We practiced naming. I pointed at **S**. “Planet, rose, winds, volcanoes,” I recited. At **T**: “Sweep, screen, water, polish, uproot.” At **Time**: “Attention, ritual, minutes.” At **F**: “Choose from circumstance.” At **C**: “Add them up.” At **G**: “What love asks.” At **α**: “The promise that doing will be enough.”

(Parenthetical code, condensed card)

$S, T, \text{Time}; F: S \rightarrow T; C: T \rightarrow \text{Time}; G: S \rightarrow \text{Time}$

$\alpha : C \circ F \Rightarrow G \quad \text{-- witness}$

$P := \Sigma s, t. \text{Id_Time}(C\ t, G\ s)$

$\phi(s) := (s, F\ s, \alpha_s)$

“Good,” he said. “Now when we forget, we can remember where to look.”

“What about the rose?” I asked, because the square had made me brave.

He drew a tiny curve in the empty space above **S** and shaded it the way people shade a word when they can’t bear to say it twice. “She is... complicated,” he said. “Her words are not always the same as her meaning. But the diagram sees her exactly: if I come at four, she becomes happy at three. If I do not come, the inequality reverses.”

(Parenthetical code)

Ritual enrichment: replace Time by Joy with order $\leq$ Joy.

Witness becomes $\alpha^\diamond : \text{Joy}(C(F(s))) \leq \text{Joy}(G(s))$.

Interpretation: rituals transport care into joy.

We left the figure there for the wind to copy. The desert, which is nothing if not rigorous, erased it slowly until even the hatch marks had become a general smoothness. But I could still see the square against the inside of my eyelids when I closed them.

That night, I dreamed of a lamp being switched on and off at perfect intervals, an island of law floating on a shrinking day. When I woke, the boy was already awake, tracing an invisible diagonal through the air with one finger, smiling at something only he could read.

“Today,” he said, “we will draw the other face. The one where the trees try to become monsters.”

“The baobabs,” I said.

He nodded. “They are only tragedies when you stop drawing.”

3) Baobabs at Dawn

Dawn braided the cold into the sand and left it there. The boy woke like a bell—no residue of sleep, only a clear note. He knelt and drew the square again, this time turning it so a new face could look at us: **Structure** and **Tasks** angled toward a fourth corner he named **Resources**.

“The baobabs,” he said, “are small at first. If you see them when they are very young, you can pull them up with your fingers. But if you forget, their roots find the planet’s seams and pry.”

He pinched the air as if plucking threads from a sleeve. “You must do it *every* morning.”

(Parenthetical code)

R : Type -- Resources

channel : R = {attention_slots, hand_strength, tool_edges, safe_surface}

A : S -> R -- available capacity from structure

D : T -> R -- demanded capacity from chosen tasks

He set the diagonal **F** again from **Structure** to **Tasks**, but now he traced the *side* that runs from **Tasks** to **Resources** and wrote **D**, and the side from **Structure** to **Resources** and wrote **A**. Between them he hatched the second face.

(Parenthetical code)

$\rho : D \circ F \leq A$ -- resource witness (lax)

Expanded: $\rho_s : D(F(s)) \leq A(s)$ componentwise in the resource poset.

“In my planet’s morning,” he said, “I have five attention slots before the sun is too high. My hands can pull twenty seedlings if the roots are fresh. The edge of my hoe is good for ten more if any are stubborn. And there is only so much *safe surface* to disturb without hurting my rose.”

He wrote in the sand as he spoke, numbers softened by wind:

(Parenthetical code)

$A(s) = (\text{slots}=5, \text{pulls}=20, \text{chops}=10, \text{safe}=12)$

If $F(s)$ yields tasks [Uproot $\times n$, Water, Sweep, Screen], then

$D(F(s)) = (\text{slots} = 1 + n/5, \text{pulls} = n, \text{chops} = \text{ceil}(n/2), \text{safe} = n)$

(toy model: one slot for non-uprooting + 1 per five uproots; two tough seedlings per hoe chop; each up-root touches one unit of safe_surface)

“Today,” he said, looking toward an invisible east, “I think there are eight.” He touched the ground with two fingers, as if greeting the enemy with courtesy.

(Parenthetical code)

Example: $n=8$

$D(F(s)) = (\text{slots}=1 + 8/5 = 2.6, \text{pulls}=8, \text{chops}=4, \text{safe}=8)$

Compare to $A(s) = (5, 20, 10, 12)$

Hence ρ_s holds: $2.6 \leq 5, 8 \leq 20, 4 \leq 10, 8 \leq 12$.

He smiled—not with triumph, but with the quiet of a face filled. “If I do not prove *this*,” he said, tapping the hatched resource face, “the time face is a dream. The volcanoes love sweeping, but they will not argue with a baobab root.”

I watched him root out imaginary seedlings with small, exact motions. A few he mimed snapping with the hoe. Each action seemed to slip back into the diagram, like a bead returning to a thread.

“What happens if you miss a day?” I asked.

He made a small, sad mouth. “Then they rewrite themselves.”

(Parenthetical code)

Rewrite system on seedlings:

seed --(missed_morning)--> sapling --(missed_morning)--> young_tree --(missed)--> BAOBAB

Monotone growth: demands escalate:

$D(\text{Uproot}(\text{seed})) \ll D(\text{Uproot}(\text{sapling})) \ll D(\text{Cut}(\text{young_tree})) \ll$

$D(\text{Contain}(\text{BAOBAB}))$

Resource inequality may fail: $D(F(s)) \leq A(s)$ not provable for larger forms.

He drew a second set of numbers, heavier this time.

(Parenthetical code)

Missed day: $n=8$ split to 6 seeds + 2 saplings

Model: pulls = $6 + 2*5 = 16$, chops = $0 + 2*3 = 6$, slots = $1 + 6/5 + 2*1 = 4.2$, safe = $6 + 2*2 = 10$

Compare to $A(s) = (5, 20, 10, 12) \rightarrow$ still holds.

Two missed days: some become young_tree:

pulls = $4 + 2*5 + 2*12 = 38 (>20) \rightarrow \rho_s$ fails on pulls.

Resource face unfilled $\Rightarrow$ the diagonal F no longer lifts to P_{res} .

He looked at me with the frankness children reserve for fatal truths. “If you let them become trees, you need neighbors.”

“And you have none,” I said.

“Not at that scale,” he admitted.

We renegotiated the morning. “What if,” I suggested, “we change **F** so that uprooting comes first, before sweeping volcanoes?”

He brightened. “Yes. And if there are more seedlings than slots, I can ask the wind to wait. She sometimes does.”

(Parenthetical code)

Refine diagonal: $F' : S \rightarrow T$ prioritizes Uroot^* until p holds.

Monotonicity: if D and A are functorial/monotone under this reorder, and A unchanged, then $p' : D(F'(s)) \leq A(s)$ may be recovered.

Transport: naturality square updates the time face later.

He added a small box near **Resources** and wrote **cap** in it, as if planting a seed word.

“This is the map that turns resources into time,” he said. “If I have ten good pulls and five attention slots, how much time can I promise? It is not exactly the same as counting tasks; sometimes pulls and slots overlap with watering.”

He drew a light arrow from **Resources** to **Time**.

(Parenthetical code)

$\text{cap} : R \rightarrow \text{Time}$ (monotone; often subadditive)

Examples:

$\text{cap}(\text{slots}, \text{pulls}, \text{chops}, \text{safe}) := a * \text{slots} + b * \text{ceil}(\text{pulls}/\kappa) + c * \text{ceil}(\text{chops}/\lambda)$

or bottleneck: $\text{cap} := \max\{a * \text{slots}, b * \text{pulls}/\kappa, c * \text{chops}/\lambda\}$

Monotone: if $r_1 \leq r_2$ then $\text{cap}(r_1) \leq \text{cap}(r_2)$.

“We will use it later,” he said, “to make the two faces agree.” He touched both hatched squares—the time face we had drawn yesterday and the resource face now. “But first you must see that this face can fail honestly.”

He erased the eight and wrote **many**. The word looked larger than the planet. “Suppose the wind buried the seedlings, and I forgot two mornings. My hands have limits even when my love does not.”

(Parenthetical code)

Failure scenario: for some s , $A(s)$ fixed, $D(F(s)) \not\leq A(s)$; no p_s .

Remedies:

(1) Change F (prioritize, call neighbors, split tasks across days): introduce F' and a *movie collar* later.

(2) Change S (add tool, sharpen edge): update $A(s)$ via $S \rightarrow S'$.

(3) Admit non-feasibility: mark face unfilled and record $\text{gap_R} := D(F(s)) - A(s)$ by channel.

He looked ashamed at the thought of limits. I, who had landed a machine in a desert with no map and called it courage, recognized the heavier brave: to measure honestly.

"You can write the gap," I said. "And then sharpen the hoe."

He nodded. "Sometimes," he confessed, "I add fake tasks. 'Straighten the pebbles,' I say, or 'polish the rake.' They make me feel industrious. But the face does not care about feelings."

(Parenthetical code)

Noise tasks: $T_{\text{noise}} \subset T$ with $D(\text{noise}) > 0$, $C(\text{noise}) > 0$, zero effect on $A(s)$ or $G(s)$
 $\Rightarrow$ harms both faces; remove by a refactor $F \rightarrow F \setminus \text{noise}$.

We made a small ledger beneath the square, a kindness to future selves:

(Parenthetical code, resource ledger)

channels: attention_slots, pulls, chops, safe_surface

$A(s)$: 5, 20, 10, 12

$D(F(s))$: 2.6, 8, 4, 8 $\rightarrow$ OK

$D(F'(s))$: 3.0, 8, 4, 8 $\rightarrow$ still OK (priority adds slot overhead)

$\text{gap_R} := \max(0, D - A)$ by channel $\rightarrow$ all zeros today

He seemed relieved by zeros the way some people are relieved by rain. "Now if I forget," he said, "I will know how much forgetting costs, and where."

He pointed to **cap** again, the way a gardener checks a sapling. "Tomorrow," he promised, "we will show that the time I swear and the resources I prove are the same oath."

"Ø," I said without meaning to.

He laughed at the Greekness of it. “Yes. The invisible knot in the middle.”

Before we left the drawing, he pressed one fingertip to each corner and spoke their names as blessings: “Structure,” “Tasks,” “Resources,” “Time.” Then he stood, brushed sand from his knees, and walked to where the day had put the seedlings he alone could see.

He pinched the air eight times. The wind approved. My throat hurt with wanting water, but my chest felt irrigated by something more practical: inequalities that hold, faces that fill, a morning that doesn’t need heroics—only exactness.

(Parenthetical code, condensed card for the second face)

$S, T, R; F: S \rightarrow T; D: T \rightarrow R; A: S \rightarrow R$

$\rho : D \circ F \leq A$ -- resource witness (componentwise)

$\text{cap} : R \rightarrow \text{Time}$ (monotone)

-- to be used next: θ compares α with $\text{cap} \circ \rho$ (via $\gamma : G \sim \text{cap} \circ A$)

We returned to the wreckage-shade with small triumphs in our pockets: eight less-than signs, a sharpened hoe, a little word—*cap*—that looked like a promise folded once.

“Tomorrow,” he said, settling by the box with the sheep, “we will build a cube. It needs only one more face, and then a place in the middle to live.”

4) A Cube for Coherence

The next morning we drew not a square but a cube—edges in the sand, corners pricked with the stick until they remembered they were points. Two faces we already knew by heart: the **time face** (Structure $\rightarrow$ Tasks $\rightarrow$ Time, with its witness that doing enough becomes being enough) and the **resource face** (Structure $\rightarrow$ Tasks $\rightarrow$ Resources, with its witness that demands fit inside capacities). The boy set them as adjacent faces of the cube, sharing the same diagonal choice **F**.

“Now,” he said, “we must show that the stories are not two stories at all.”

He planted a small arrow from **Resources** to **Time** and labeled it **cap**. “This is how capacities speak minutes,” he said. “It’s the map that turns pulls and slots into time that can be promised.”

(Parenthetical code)

$\text{cap} : R \rightarrow \text{Time}$ (monotone; usually subadditive or bottleneck)

Then he drew a faint thread from the **Structure**→**Time** edge to the composite **Structure**→**Resources**→**Time** and wrote γ beside it, a modest letter for a steady bridge. “This is how my world’s demand and my capacity-of-resources compare,” he said. “Sometimes they are the same, sometimes one is gentler.”

(Parenthetical code)

$\gamma_s : \text{Id_Time}(G(s), \text{cap}(A(s)))$ (*strict*)

or a lax comparison $\gamma_s : G(s) \geq \text{cap}(A(s))$ (*predictor looser than capacity*)

He tapped the cube’s inside. “The knot belongs here,” he said, and with the stick he traced a little tetrahedron inside the cube, its points touching the three operational corners and the one we could not name without smiling: the **process** corner.

“Two faces are filled,” he said. “But the middle must promise that the time I count directly equals the time my capacities imply.”

He wrote the pieces in the sand like ingredients:

(Parenthetical code, faces)

Time-face: $\alpha_s : C(F(s)) \leq G(s)$

Resource-face: $\rho_s : D(F(s)) \leq A(s)$

Capacity map: $\text{cap} : R \rightarrow \text{Time} \rightarrow \text{cap}_*(\rho_s) : \text{cap}(D(F(s))) \leq \text{cap}(A(s))$

Bridge in Time: $\text{bridge}_s := [C(F(s)) \leq \text{cap}(D(F(s))) \leq \text{cap}(A(s))] ; \gamma_s$

He lifted his finger and pressed it into the center. “And here is the coherence.”

(Parenthetical code, interior 3-cell)

$\theta_s : \text{Id}_{\{\text{Comp_Time}\}}(\alpha_s, \text{bridge}_s)$

(*in strict settings: a 3-path equating two 2-paths; in lax: a certificate that the two inequality chains are the same/refinement*)

We tested it on the numbers we'd whispered into yesterday's square. "Suppose," I said, "your morning costs thirty-five minutes by counting tasks."

He nodded. "And the planet asks forty. So $\alpha_s : 35 \leq 40$."

"We measured capacities as $A(s) = (\text{slots}=5, \text{pulls}=20, \text{chops}=10, \text{safe}=12)$. Then $\text{cap}(A(s)) = 40$ min by your conversion." He smiled; it pleased him when conversions behaved. "Your chosen routine demanded $D(F(s)) = (2.6, 8, 4, 8)$, so $\text{cap}(D(F(s))) = 35$ min. The bridge says $35 \leq 40$, same as α_s ."

He drew a tiny equals sign between the two arrows. " θ_s ," he said, as if naming a bird in the wild.

(Parenthetical code, plugged example)

$C(F(s))=35, G(s)=40, \text{cap}(D(F(s)))=35, \text{cap}(A(s))=40$

$\alpha_s : 35 \leq 40$

$\text{cap}_*(\rho_s) : 35 \leq 40$

$\gamma_s : 40=40$ (strict)

θ_s : identifies α_s with $(\text{cap} \circ \rho_s ;; \gamma_s)$

"But what if," I asked, "your capacity model is too optimistic? Suppose $\text{cap}(A(s))$ says thirty, but your counted morning always takes forty."

He pulled the cube toward him and scratched out the inside with quick, forgiving lines. "Then there is *no* θ to be had with this cap," he said. "We must learn. Either cap is missing a channel (for example, the time my eyes need to see if a seedling hides), or my conversion rates are wrong."

He added a new resource in the margin—**look_time**—and fed it to **cap** like a correction.

(Parenthetical code, model repair)

Extend R with a channel look. Update $A(s)$ and $D(F(s))$ with look counts.

Define $\text{cap}'(r) := \text{cap}(r_{\text{without_look}}) + a_{\text{look}} * r.\text{look}$.

Now recompute: seek α_s and $\text{cap}'_*(\rho_s)$ that match under $\gamma_s' : G(s) \sim \text{cap}'(A(s))$.

If matched, build new θ_s' .

“Sometimes,” he added, “ $G(s)$ is a promise I made too large. I can set γ_s to compare instead of equal: $G(s) \geq \text{cap}(A(s))$. The inside then says that the time I counted equals the capacity bridge, and both together sit under my promise. That is honest, too—if I admit it in writing.”

He shaded the inside of the cube with little dots until it looked like it was humming. “It is a happy sound,” he said. “It means the two faces are telling the same truth.”

(Parenthetical code, the full cube type)

Process_3D :=

$\Sigma s:S. \Sigma t:T.$

$(w_time : C(t) \leq G(s)) \quad \text{-- } \alpha$

$\times (w_res : D(t) \leq A(s)) \quad \text{-- } \rho$

$\times (\theta : w_time \approx (\text{cap}_*(w_res) ;; \gamma_s))$

-- ($\approx$ reads “is the same certificate as / refines to”)

We sat with the cube between us like a house built in an afternoon. A fly landed on one corner and walked its edges as if checking glue lines. The boy laughed without sound.

“Grown-ups would like a number for θ ,” I said.

He considered and then offered mercy. “We can write a *gap*, if you like—how many minutes apart the two routes are when they pretend to be the same. But the point is not the number. The point is whether the interior can be filled at all.”

(Parenthetical code, audit metric)

$\text{theta_gap}(s) := | (\alpha\text{-route}) - (\text{cap} \circ \rho + \gamma\text{-route}) |$

Target = 0 (strict coherence) or small (modeling tolerance).

If $\text{theta_gap} > 0$, mark which channel or conversion caused the discrepancy.

He folded his hands. “Now when I say, ‘I will come at four o’clock,’ I mean something you can see. I mean that α exists, and ρ exists, and θ binds them. If any

piece fails, I owe a correction—a sharper hoe, a new channel, a smaller promise, or a different routine.”

“And if all three hold?”

He smiled in a way that made the cube look less like geometry and more like a room. “Then we can go visiting.”

We left the cube in the sand, knowing the wind would file its edges and the sun would erase its faces, but also knowing we could draw it again with the same names and find the same interior waiting, like a song you can hum even after the radio is gone.

5) The King Who Commands Sunsets

We came to a planet so small it had only room for a throne and the authority to sit in it. The King wore an ermine robe that trailed off the edge of the world and fell forever. He greeted us with the relief of someone whose subject has at last arrived.

“Approach,” he said, and then, cautiously, “if it is lawful.”

“What do you command?” the boy asked, polite as weather.

“Everything,” said the King, with the modesty of inevitability. “I do not tolerate disobedience, so I command only what will happen.”

He demonstrated. He consulted an almanac, the horizon, the angle of his own shadow, and pronounced: “I order the sun to set... at seven forty.” And when the sky complied, he looked pleased but not surprised, as a mathematician is pleased when a lemma holds.

(Parenthetical code)

Acts : Cat (things that can be done)

Inev : FullSubcat(Acts) (things that will happen anyway)

i : Inev -> Acts (inclusion)

□ : Acts -> Inev (reflector)

$\Box \dashv i$ (adjunction; reflective subcategory)

Unit (law-as-decree): $\eta_a : a \rightarrow i(\Box a)$

Meaning: a “command” factors an act through the inevitable part.

The King’s laws had the softness of good gloves. He commanded the tide to rise—when the moon said yes. He commanded yawns at midnight and hunger at noon. He wrote no edicts against storms; he proclaimed the storm *recognized*.

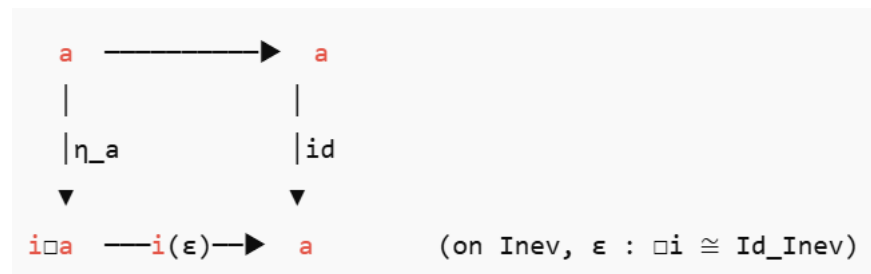
“Is there nothing you forbid?” I asked.

“I forbid nonsense,” he said, almost kindly. “An absolute monarch must be absolutely reasonable.”

The boy’s eyes asked for a diagram. I scratched a small triangle in the dust of the planet’s only plain.

(Parenthetical code)

For any $a \in \text{Acts}$, obedience is a commuting triangle:



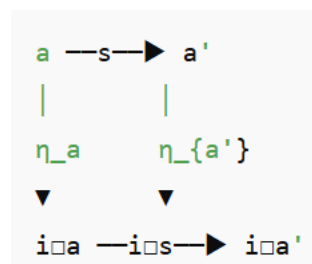
Snake identities guarantee coherence of “command then forget” and “forget then command.”

“What about sunsets?” the boy asked.

“I command them when it is time,” said the King. “I do not command them when it is not.”

(Parenthetical code)

Temporal lawfulness as a **naturality square**: for sunset morphism $s: \text{Day} \rightarrow \text{Night}$,



The decree η is natural in time; the square is filled.

“Then your power is to say ‘yes’ to what already is,” the boy said gently.

“My power,” said the King, “is to keep adults from ridiculous orders and children from despair. I align words with the world.”

He made us Ministers of Justice in a kingdom with no criminals. Our duty, he said, was to pardon. “It is a satisfying occupation,” he assured us, “if one enjoys right triangles.”

(Parenthetical code)

Authority gap metric:

$\text{gap}(a) := \text{dist}(id_a, i(\epsilon) \circ \eta_a) = 0$ by reflection.

Face coverage = 1 (all commands factor), interior trivial (coherence holds by adjunction).

When we prepared to leave, he looked stricken. “Do not go—it is not lawful.”

“Majesty,” the boy said, “we must visit a conceited man, a drunkard, a businessman, a lamplighter, and a geographer.”

“Then,” said the King, recovering, “I command you to depart... immediately.” He raised his scepter just as our feet pushed off.

As we fell toward the next star, the boy waved back, grateful for a crown that understood its own functor. “He is a good king,” he said. “He proves that to rule is to factor.”

“And to love?” I asked.

“To factor, and then to fill,” he said, watching the light slip along the curve of an unseen sea.

6) The Applause Machine

The next world had a gentleman planted on it like a flag. He wore a hat designed only for tipping. “Ah! Admirers!” he cried, though we had not yet admired. “Clap! Quickly!”

We clapped, two pairs of hands in a universe with more stars than reasons. He bowed, trembled, and brightened as if plugged into a socket.

“You admire me truly?” he said.

“What does *truly* mean?” the boy asked.

“It means: clap again.”

We clapped again. He bowed deeper, then waited, trembling, for the future to become the past once more.

(Parenthetical code)

Let X be “persons.” Define a functor that wraps anything in “praise”:

$M : X \rightarrow M, M(x) := \text{Praised}(x)$

with unit $\eta_x : x \rightarrow M(x)$ (“admire me”) and multiplication $\mu_x : M(M(x)) \rightarrow M(x)$ (“double praise collapses to praise”).

This is an **idempotent monad**: $\mu \circ M\eta = \text{id} = \mu \circ \eta M$, and $\mu \circ M\mu = \mu \circ \mu M$.

He had a single conversation, always already begun.

“Do you love me?” he asked the boy.

“I love my rose,” the boy said.

The gentleman looked injured, as if physics had failed. “Then applaud my restraint.”

We clapped, because we are soft, and because it was easy, and because the desert had taught us that there are worse noises than hands.

(Parenthetical code)

His world has only one interesting arrow $\text{applause} : x \rightarrow M(x)$.

Composition is absorbed: $\text{applause} ; \text{applause} = \eta ; \mu = \text{applause}$.

Diagrammatically, every rectangle filled by M folds into itself.

The boy tried to trade something real. “If I tell you I admire your patience,” he offered, “will you tell me what you admire?”

“I admire being admired,” said the man. It was a theorem to him, not a confession.

(Parenthetical code)

No counit $\varepsilon : M(x) \rightarrow x$ is provided (no way to “cash out” praise into substance).

Hence, there is **no map to Meaning**: given $\text{Val} : X \rightarrow \text{Joy}$, there is no natural $\tau : M \Rightarrow \text{Id}$ s.t. $\text{Val} \circ \tau = (\text{something honest})$. The **face to value** remains unfilled.

The boy bowed (he has excellent manners) and asked: “Do you ever stop?”

“When the planet sleeps,” said the man, “I sleep. When it wakes, I tip my hat to it. We go on like this, the world and I, forever.”

(Parenthetical code)

Loop invariant: a fixed point of M : $\text{fix} := x$ with $x \cong M(x)$.

But the Lawvere metric on Joy does not strictly increase:

$\text{Joy}(x) = \text{Joy}(M(x))$ (no progress).

A movie of this scene is constant: $u \dashv \rightarrow x$, faces filled solely by η, μ , no interior θ to bind to an external cap or goal.

“Thank you,” the boy said, and I recognized mercy.

We clapped once more (for travel’s sake) and departed while he was mid-bow, which felt like leaving a metronome on a table.

In the air between planets, the boy considered what our hands had purchased. “It is nice to please someone,” he said, “but I could not live on it.”

“What do you live on?” I asked.

“On witnesses,” he said simply. “On the places where doing and being agree.”

7) A Bottle Without Bottom

The next planet smelled like the inside of a locked cupboard. Bottles stood in phalanxes: ranks of green, clear, brown—each with a story it would not tell. On a stool a man sat bent around a tiny circle: bottle → glass → mouth → table → sigh.

“What are you doing?” the boy asked, though he already knew.

“Drinking,” said the man, without weather in his voice.

“Why do you drink?”

“To forget,” he said.

“To forget what?”

“That I am ashamed.”

“Ashamed of what?”

“Of drinking,” he said, and looking up hurt him.

The circle tightened. We had found a proof without a statement.

(Parenthetical code)

State space $X = \{\text{Sober}, \text{Drinking}, \text{Ashamed}\}$ with rewrite rules:

r1: Sober → Drinking

r2: Drinking → Ashamed

r3: Ashamed → Drinking

No normal form; infinite loop: Sober → Drinking → Ashamed → Drinking → ...

No terminating measure $\mu : X \rightarrow \mathbb{N}$ s.t. each step decreases μ .

Hence **Newman’s lemma** (terminating + local confluence $\Rightarrow$ confluence) does not apply.

The confluence diamond is *not* fillable; the face stays open.

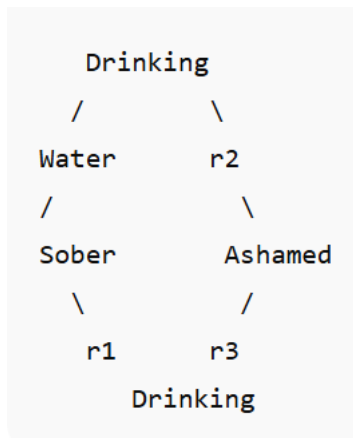
The boy tried to place a hand on an edge somewhere. “If I bring water, will it help?”

“For a moment,” said the man. “But then I will be ashamed of thirst.”

(Parenthetical code)

Adding a task Water gives a new rule Water: Drinking $\rightarrow$ Sober, but rule r1 reopens the loop.

Critical pair with r3:



Local diamond exists but does not terminate $\Rightarrow$ global confluence still fails.

The boy looked at his feet the way you look at a map that refuses an answer. “Do you have a morning?” he asked. “A square you draw and fill?”

“I have a clock,” said the man, “so I may be on time for not being on time.”

He lifted the glass. The world climbed up toward him and fell back exhausted.

(Parenthetical code)

Resource/time faces cannot be certified:

- **Time face** fails: there is no $\alpha : C \circ F \Rightarrow G$ with G a meaningful ritual demand; $C(F(s))$ accumulates cycles with no witness that “doing becomes being.”
- **Resource face** is parasitic: $D(F(s))$ consumes $A(s)$ (time, liver, coin) without reducing any goal metric; no θ exists under any sensible $\text{cap} : R \rightarrow \text{Time}$ enriched by Joy.

“But you could be happy,” the boy said, not as a threat but as a theorem he wanted to hold.

The man winced. “Happiness is loud.”

(Parenthetical code)

Joy-enrichment:

Joy : Poset, with projection $\text{Val} : X \rightarrow \text{Joy}$.

Along the loop, Val is non-increasing or oscillatory; there is no monotone descent to a fixed point.

A *movie collar* cannot legalize a jump to wellness without an external equivalence (doctor, friend) supplying a new rule—an extra edge not present here.

We waited. He did not ask us to stay, and so we stayed a respectful minute.

“I’m sorry,” the boy said at last, because sorrow is the shape of not being able to draw a face where one is needed.

“Me too,” said the man, and put his head down on the table the way some put their heads on rails.

(Parenthetical code, diagnosis card)

Rewrite system: non-terminating cycle

No global confluence

No Process_{2D}, Process_{3D} witnesses

θ : n/a

Remedy (external): add new rule with descent metric; otherwise, mark face unfilled and move on

We left with the softest possible footsteps. In the air between worlds the boy pressed his knuckle to his lip and kept it there, as if to stop a word from escaping and breaking on the floor.

“Sometimes,” he said, “the proof we can give is to walk away without noise.”

8) The Star Accountant

The next planet glittered like a coin and felt like a desk. A man sat behind an adding machine whose ribbon trailed into the future. He did not look up when we arrived; he was counting light.

“One, two, three, five hundred one million, six hundred twenty-two thousand, seven hundred thirty-one,” he said, and made a small church of his fingers over the number, as if to keep it from running away.

“What are you doing?” the boy asked.

“I am a serious man,” said the man. “I count the stars. I possess them.”

“How can you *possess* the stars?” the boy asked, astonished.

“By writing the numbers in my ledger,” the man said. “If you find a diamond, it belongs to you. If you find an island that belongs to nobody, it is yours. I possess the stars because I have thought of it first.”

He lifted a stack of papers as if it were a baby and patted it into obedience.

(Parenthetical code)

Let Stars be a very large set. Define a counting functor

Count : Stars $\rightarrow$ N (or to N^k for partitions).

Define a “ledger object” as a **commutative monoid** $(L, +, 0)$ with an inclusion $\iota : N \rightarrow L$.

Ownership claim: a natural transformation does *not* exist from actual star-like *meaning* to L. There is no section $\text{sec} : L \rightarrow \text{Stars}$ and no semantics map $\text{Mean} : \text{Stars} \rightarrow \text{Joy}$ that factors through L.

The boy tried to be courteous. “What do you *do* with the stars?” he asked.

“Do?” The man blinked, confused by verbs that touch the world. “I own them. I put them in the bank.”

“How?”

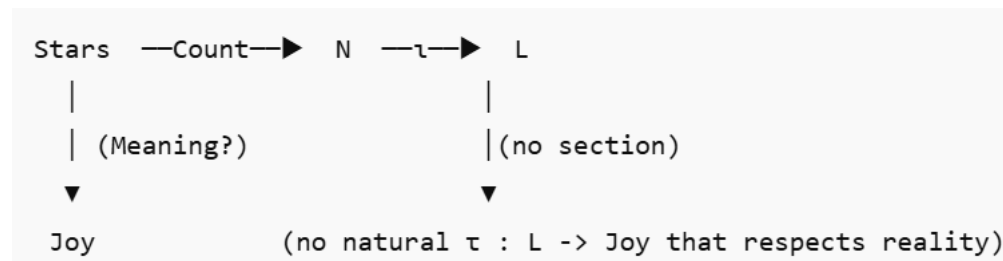
“I write the number of my stars on a little paper. Then I lock the paper in a drawer.”

“That is all?”

“That is enough,” said the man, and smiled the way a padlock might if it could.

(Parenthetical code)

We can view L as a **decoupled structure**: a monoid of records with no functor back to phenomena:



Thus the **interior is hollow**: there is no θ equating “ledger time” with any $\text{cap}(\cdot)$ from real resources; the face to value/meaning is unfilled.

“To own a flower,” the boy said carefully, “is to water it.”

“To own a volcano,” I added, “is to sweep it.”

“To own a star,” the man said, “is to number it.”

The boy looked hurt, the way numbers sometimes hurt when they arrive empty.

(Parenthetical code)

Try to draw the process square:

- $S = \text{Stars}$, $T = \text{Tasks}$ (e.g., Explore, Chart, Care), $\text{Time} = \text{MeaningfulAttention}$.
- The businessman’s diagonal is $F_{\text{biz}} : S \rightarrow T_{\text{biz}}$ with $T_{\text{biz}} = \{\text{WriteLedger}\}$.
- $C : T_{\text{biz}} \rightarrow \text{Time}$ is constant near 0 (ledger-writing yields no care for stars). Hence the **time face** $\alpha : C \circ F_{\text{biz}} \Rightarrow G$ fails for any G representing stewardship.

The **resource face** $\rho : D \circ F_{\text{biz}} \leq A$ is trivial (paper and ink) and irrelevant.

No θ can bind “ledger-time” to a capacity model $\text{cap} : R \rightarrow \text{Time}$ that touches stars.

“What makes a star *yours*?” the boy asked. “Is there a witness?”

The man spread his hands over the ledger. "This."

(Parenthetical code)

He supplies an *idempotent endomorphism* on L : $\sigma : L \rightarrow L$ ("reconcile books"), $\sigma \circ \sigma = \sigma$.

But there is no **universal property** that touches stars: no pullback/comma object that contains (star, task, witness) with a meaningful `Id_Time` filler.

"You are not useful to the stars," the boy said at last. It was not an insult; it was a lemma.

The man returned to his adding machine, injured only where numbers feel.

As we left, the boy folded his eyebrows into worry. "It is easy to be serious without being responsible," he said. "You can fill a ledger without filling a face."

I nodded, thinking of my toolbox. "You can also fix a plane without fixing your life."

He smiled at me sideways, which meant he forgave me in advance. "Then we should visit someone whose law holds even when his day shrinks," he said. "To remember that exactness is not the same as emptiness."

9) The Lamplighter's Day

This planet turned like a bead on a hot wire. A single streetlamp stood there, perfectly unnecessary and therefore sacred. Its keeper wore a cap with the dignity of a clock.

"Good morning," said the boy.

"It isn't," the lamplighter said, and lit the lamp. "Now it is."

He waited a breath, then blew it out. "And now it is night."

"Why do you light and put out so often?" I asked.

"Orders," he said simply. "When I started, the day was long. The planet has been spinning quicker ever since. A minute here is a second there now. But the rule has not changed: 'Light at night. Extinguish by day.' So I obey."

(Parenthetical code)

$S := \{\delta : \text{DayLength}, \text{rule} : \text{"toggle each night/day boundary"}\}$

$T := \{\text{toggle}\}^*$ (words made of toggle)

$\text{Time} := (\mathbb{N}, +, 0, \leq)$ (discrete steps)

$F_\delta(s) := \text{toggle}^n(\delta)$ (how many toggles per full rotation)

$C(\text{toggle}^n) = n$ (monoidal length)

$G(s) = 2 * \text{turns}(s)$ (one on, one off per boundary; here turns = 1 $\Rightarrow G = 2/\text{rotation}$, scaled to unit period)

He showed us his ledger: a thin column of integers becoming thick. He had crossed out sleep and written duty in its place.

"The law is clean," he said, lighting, then darkening. "It is not the law's fault that the day shrinks."

(Parenthetical code)

Time face (often strict):

$\alpha_\delta : C(F_\delta(s)) = G(s)$ (exactly two toggles per rotation, regardless of δ)

Resource face:

Let $R := \{\text{waking_minutes}, \text{wrist_strength}, \text{matches}\}$ with

$A_\delta(s) = (\text{waking} = \delta, \text{wrist} = \infty, \text{matches} = \infty)$ (idealized)

$D(F_\delta(s)) = (\text{waking} = C(F_\delta(s)) * \epsilon, \dots)$

For small enough δ , $D(F_\delta(s)) \leq A_\delta(s)$; as $\delta \downarrow 0$, eventually $D > A \Rightarrow \rho$ fails.

He smiled at the lamp as one smiles at a partner in an old dance. He did not complain. He shortened his meals until they fit between sparks.

(Parenthetical code)

Movie in dHoTT: parameterize by $u \in [0,1]$ with $\delta(u) \downarrow$.

Frames:

$\text{Process_2D}(u) = \sum s_{u,t}. \text{Id_Time}(C_u(t), G_u(s))$ still holds (two toggles).

$\text{Process_3D}(u)$ holds until $u = u^*$ where waking budget fails; there, resource face unfilled; insert a **collar** if the rule is amended.

The boy watched the cadence: duty, rest; rest, duty—an honest pendulum. "Your law is beautiful," he said.

“It is all I have,” said the lamplighter, and lit the lamp again with a tenderness that made the flame look like a witness.

(Parenthetical code)

Invariants:

$\text{slack_time}(u) := G_u(s) - C_u(F_u(s)) = 0$ (strict equality)

$\text{res_util}(u) := D(F_u(s)) / A_u(s)$ increases to 1 and then >1

$\text{theta_gap}(u) = 0$ while p holds; undefined past u^* unless rule changes.

“What would you do,” I asked, “if the day vanished?”

“Ask for lawful mercy,” he said. “Change the rule to ‘auto-lamp,’ and keep watch over meaning instead.”

He looked at us carefully, then at the lamp, then at the small horizon that circled like a ring tossed in air. “But until they change the words, I will keep them.”

(Parenthetical code)

Lawful refactor (collar):

Old rule Rule_0 : toggle@boundaries. New rule Rule_1 : auto@boundaries, human@Sabbath.

Provide equivalence $\varepsilon : C_0 \Rightarrow C_1$ and amend G accordingly.

Insert collar frames carrying the natural iso; rebuild α , ρ , θ under Rule_1 .

We left him mid-breath between night and day, which is where all vows are kept. The boy waved each time the light returned. The lamplighter counted our waves as if they were seconds.

“Exactness,” the boy said when we were back among the dark, “is a kind of love.” He glanced at me as if to add: provided there is a face to fill.

10) The Geographer and the Map

We landed on a planet that was mostly a book. The Geographer wore spectacles that kept his eyes from leaking into the world. He welcomed us without standing, as if standing were a rumor.

“I am a scholar,” he said. “I record what *is*.”

“Do you go and see?” the boy asked.

“I do not,” said the Geographer, proud of his abstinence. “I interrogate explorers. It preserves objectivity.”

He opened an atlas the size of a window. Mountains in clean ink, rivers with obedient bends, deserts like elegant omissions. “We write only the *eternal*,” he said. “We do not record things that pass.”

“My flower?” the boy asked, already bracing.

“Ephemeral,” said the Geographer, and the word fell like a lid.

(Parenthetical code)

Let Places be a small category of regions; let $\text{Obs} : \text{Places}^{\text{op}} \rightarrow \text{Data}$ be a presheaf of **stable** observations (mountains, rivers).

Include $\text{incl} : \text{Stable} \hookrightarrow \text{All}$ as a subcategory embedding.

The Geographer computes a **right Kan extension** along incl to promote stable observations to global records:

$\text{Atlas} := \text{Ran}_{\{\text{incl}\}} (\text{Obs} \circ \text{incl})$

Right Kan = limit-like: preserves products/limits, favors permanence.

Ephemeral features (flowers, foxes, friendships) live in a presheaf $\text{Eph} : \text{Places}^{\text{op}} \rightarrow \text{Data}$ not in the image of incl , so no component of Atlas sees them.

He showed us how to write a mountain. “An explorer reports: ‘Granite, 3,712 meters.’ I demand proofs: samples, triangulation, second parties.” He seemed kind, the way a gate is kind: impersonal, consistent.

“And if someone tells you, ‘There is a rose that coughs when the wind is cruel, but sings when you come at four o’clock’?”

He adjusted his spectacles. “We do not note *roses*. We note *gardens*. We do not note *songs*. We note *climates*.”

(Parenthetical code)

Two functors:

Stable : World $\rightarrow$ Data (geologic)

Tender : World $\rightarrow$ Joy (ritual/heart)

Geographer keeps only Stable. There is no natural transformation $\tau : \text{Tender} \Rightarrow \text{Stable}$ —categories differ.

Thus, any **process square** whose G demands ritual for a rose has no factorization through Atlas; no $\alpha : C \circ F \Rightarrow G$ with G sourced from Atlas. The **face to love** cannot be filled by these records.

The boy looked at the elegant rivers as if they'd been laundered too many times. "Where would you send an explorer now?" he asked.

"Earth," said the Geographer, with the relief of a man who can finally close a drawer.

He sharpened a pencil until it became a spear and wrote our departure in a margin so thin it was almost the future. "Beware of oceans," he added, as if warning us about commas.

(Parenthetical code)

Map-territory card:

incl: Stable $\hookrightarrow$ All

Obs_stable : Stable^{op} $\rightarrow$ Data

Atlas := Ran_{incl} Obs_stable -- records only what admits limits

Eph not preserved $\Rightarrow$ roses invisible

Result: there is no $\gamma : G \sim \text{cap}(A)$ sourced from Atlas for Tender/ritual G

Outside, the boy closed his eyes and saw his flower: not granite, not datum, but a path in time that had learned his footsteps.

"Maps are good," he said softly. "But we must not mistake their limits for our own."

We left the planet of books for the place that makes them. Earth rolled up toward us like a proof waiting for its witness.

11) A Garden and a Proof

Earth rose under us like an answered question. We walked until the air smelled of green. Then we stopped: a thousand roses lifted their faces, each as precise as a compass, each as indifferent as a clock.

The boy stared as if struck from within. “They are all... like her.”

I felt the weight of his next breath before it happened. He sat down in the middle of the garden and cried the way sand accepts rain—quietly, completely.

“You are not *her*,” he told the roses, but the roses only arranged their petals into very good arguments.

(Parenthetical code)

Garden : Fin n $\rightarrow$ Rose (a finite family)

There is a **shape equivalence** among many Garden(i)—same species, color, form. But *identity* in HoTT is a **path** (witness), not mere equivalence of presentations.

I sat beside him because that is what you do when no lemma will fit. Wind wrote small proofs in the grass and let them go unread.

“They are beautiful,” he said at last, “but they are not mine.”

“What makes one *yours*?” I asked, not idly.

He looked at his hands the way a book looks at its margin notes. “Time,” he said. “Attention. Promises. The way four o’clock starts at three.”

(Parenthetical code)

Define the **care functional**:

CareSpent : Rose $\rightarrow$ Time (minutes-of-ritual actually given)

Fix today’s structure $s : S$ (planet + weather + needs).

Then the *type of a bond* is

MyRose :=

$\Sigma r:\text{Rose}.$

$\text{Id_Time}(\text{CareSpent}(r), G(s))$

A pair (r, w) with $w : \text{Id_Time}(\text{CareSpent}(r), G(s))$ is a **witness** that this rose is the one whose care matches the demanded ritual $G(s)$.

He stood, brushed the garden from his knees, and walked between the rows. He did not touch any flower. Instead, he counted steps softly, like beads: watering days, glass-polishing days, wind-screen days, volcano-sweeping days. When he reached the end, his shoulders had found their proper height again.

“They are beautiful,” he repeated, and now the sentence held.

(Parenthetical code, univalence-flavored aside)

There may be an **equivalence** of shapes

$e : \text{Equiv}(\text{Garden}(i), \text{Garden}(j))$,

but **identity** demands a path in Time: a witness $w_{ij} : \text{Id_Time}(\text{CareSpent}(\text{Garden}(i)), \text{CareSpent}(\text{Garden}(j)))$.

Only one index i^* admits a path to $G(s)$; that one is “his.”

He knelt to the nearest blossom and bowed—a ceremony as old as hands. “You are like my rose,” he said. “But you are not my rose.”

The roses listened like judges: impartial, exact, uninterested in grief.

(Parenthetical code, tiny worked instance)

Let $\text{Time} = (\mathbb{N}, +, 0, \leq)$ minutes. Suppose over 10 days:

$\text{CareSpent}(\text{Garden}(1)) = 0$

$\text{CareSpent}(\text{Garden}(2)) = 17$

...

$\text{CareSpent}(\text{Garden}(k)) = 400$

With $G(s) = 400$ (the ritual demand), we can produce exactly one witness $w : \text{Id_Time}(400, 400)$ (strict) or $w : 400 \leq 400$ (lax).
Thus $\text{MyRose} \cong \{ (\text{Garden}(k), w) \}$ is **contractible**—a single inhabitant up to path.

“Why does it hurt?” he asked, already knowing.

“Because equivalence can dress like identity from far away,” I said. “And because witnesses live near.”

He laughed a little, without sound, and wiped his face with both wrists like a fox would. “Then I will keep my witness,” he said. “If I ever forget, I will read it.”

He turned to the garden and addressed the whole discipline: “You are lovely. But I have wasted no water on you. No glass of mine shines on your heads. You do not know the sound of my steps. You are not responsible for me, and I am not responsible for you. That is why you are only beautiful.”

The garden accepted the theorem.

(Parenthetical code, transport/naturality card)

If tomorrow the structure changes $s \rightarrow s'$ (weather softens, volcanoes sleep), we **transport** the bond:

$$\text{tr} : \text{Id_S}(s, s') \rightarrow (\text{MyRose}(s) \rightarrow \text{MyRose}'(s'))$$

$$\text{tr}(p)(r, w) := (r, \text{transport_Time}(p)(w))$$

Responsibility persists by transport; *renaming* the day does not unmake the path.

He took my hand the way one takes a tool—gently, for use rather than for show.

“Now I can visit the fox,” he said. “I need a friend who understands witnesses.”

We left the garden with our pockets empty and our diagrams full. Behind us the roses resumed being correct. Ahead of us a small field waited to be turned into a square at four o'clock.

12) Taming at Four O’Clock

The fox watched us from the color of grass. When he spoke, the grass spoke with him. “Please,” he said, “tame me.”

“What does that mean?” the boy asked, though his voice already knew.

“It means ‘establish ties,’ ” said the fox. “We will need rituals. Come each day at the same hour. Sit a little closer. Do not speak at first.”

So we drew a new figure in the field: two small triangles sharing an edge and two squares between them, a little chapel of promises.

“First,” said the fox, “there is the *beginning*—when you decide to begin. That is a unit.”

“And last,” he added, “there is the *keeping*—when what we began is real. That is a counit.”

(Parenthetical code — the adjunction)

$\text{Tame} \dashv \text{Forget} \quad \text{-- categories: Wild} \leftrightarrow \text{Friend}$

$\eta_x : x \rightarrow \text{Forget}(\text{Tame}(x)) \quad \text{-- unit (“begin coming at four”)}$

$\epsilon_y : \text{Tame}(\text{Forget}(y)) \rightarrow y \quad \text{-- counit (“the bond is real now”)}$

Snake identities (two squares filled):

$U(\epsilon) ; \eta = \text{id} \quad \text{and} \quad \epsilon ; F(\eta) = \text{id}$

We built the ritual like a staircase you can climb with your heart. The first day the boy sat far, and the fox pretended not to see. The second day the boy came at the same hour, and the fox’s tail forgot to be a secret. By the third, the field had learned the sound of footsteps and sorted it from wind.

(Parenthetical code — the movie of taming)

$u \in [0,1] \quad \text{-- days normalized}$

$\text{Process}(u) :=$

$\Sigma (p:\text{Person})(f:\text{Fox}).$

$(\eta_u : \text{begin}(p,f)) \times (\epsilon_u : \text{keep}(p,f)) \times \Theta_u$

where Θ_u witnesses snake identities up to time u .

Continuity: $u \mapsto (\eta_u, \epsilon_u)$ is a path; frame-to-frame changes are higher paths; no jumps without collars.

Each afternoon became a lens: the future gathered itself at three o'clock and shone at four. We could see it even in the wheat, which began to glow with an irrelevant gold—the color of the boy's hair.

(Parenthetical code — enrichment)

Replace Time by Joy (a poset), and lift the witness:

$J : \text{Time} \rightarrow \text{Joy} \quad \text{-- functor mapping minutes to value}$

$\alpha^\diamond : J(C(F(s))) \leq_{\text{Joy}} J(G(s))$

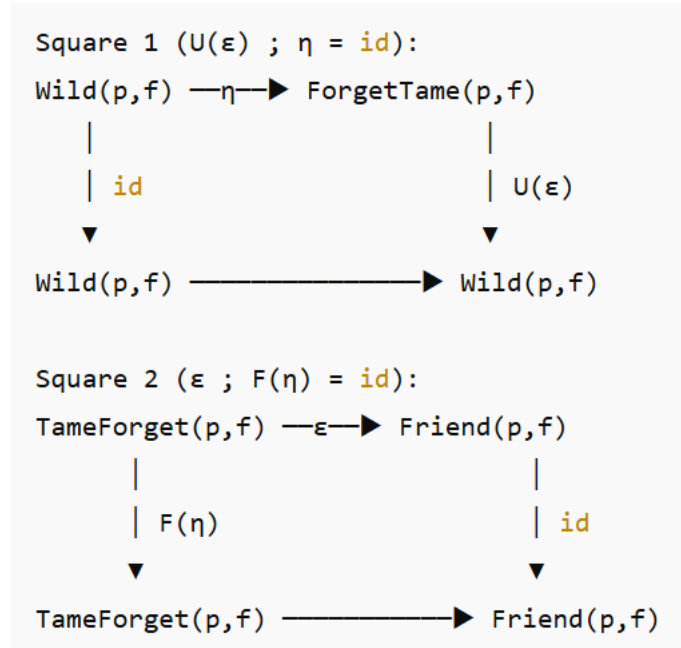
Rituals transport time into joy; the inequality strengthens frame by frame.

“Do you need me to bring gifts?” the boy asked.

“Bring exactness,” said the fox. “It is enough.”

We traced two little squares—the **snake identities**—and filled them with a grainy patience. The fox explained that promises should commute with forgetting and keeping: if you unmake a bond, you must land where you began; if you keep a bond, you must not have to begin again.

(Parenthetical code — the two squares, drawn as faces)



Both faces are **filled**: lawful taming is reversible in the right ways.

By the seventh day the fox's silence had changed its species. It was no longer fear; it was listening. "It would be better," he said, "if you came at four. Then at three I shall begin to be happy."

The boy laughed, because he had been happy since breakfast. "Then at two I shall be impatient," he said, "and at one I shall be impossible." Rituals were making time into a shape they could hold.

(Parenthetical code — the bond as a type)

$\text{Bond}(p,f) :=$

$\Sigma (\eta, \epsilon, \Theta). (\text{snake identities hold}) \times$

$\Sigma w_Time : \text{Id_Time}(\text{CareSpent}(p,f), G(s))$

$\Sigma w_Joy : \text{Id_Joy}(\text{Joy}(\text{CareSpent}), \text{Joy}(\text{DemandedRitual}))$

Responsibility = carrying $(\eta, \epsilon, \Theta, w_Time, w_Joy)$ forward under change.

“Now,” said the fox, “something essential will become invisible.” He did not mean hidden; he meant stored in a witness.

We wrote a small equality between wheat and laughter.

(Parenthetical code — transport to meanings)

Wheat : World $\rightarrow$ Joy

Laughter : Stars $\rightarrow$ Joy

Given $\text{Bond}(p, f)$, we obtain natural maps:

Wheat $\circ$ Nearness(p) $\Rightarrow$ Joy

Laughter $\circ$ Remember(p) $\Rightarrow$ Joy

The **coherence** is that both routes to Joy agree *because* of the bond; a θ in Joy equates them.

“Will you cry?” the boy asked, because it is best to declare endpoints.

“Yes,” said the fox, and he smiled the smile that proofs wear when they finally commute. “But I will also see wheat.”

The day we had to leave, we built a **collar**—a tiny movie that legalized the jump from “together” to “responsibly apart.” The boy came at four one last time and sat so close the diagram could have been drawn on a single grain.

(Parenthetical code — collar for parting)

At u^* :

old maps ($\eta_{\text{old}}, \varepsilon_{\text{old}}$) $\simeq$ new maps ($\eta_{\text{new}}=\emptyset, \varepsilon_{\text{new}}=\text{remembrance}$)

Provide equivalence data E and a collar $u \in [u^* - \varepsilon, u^* + \varepsilon]$

Transport Bond via E ; produce remembrance witnesses R

The snake squares remain filled in the *new* category (memory instead of presence).

Before we went, the fox gave his theorem. “You become responsible, forever, for what you have tamed.”

We wrote it on the inside of the boy’s pocket, where hands go in winter.

(Parenthetical code — persistence law)

Responsibility is **stable under transport**: for any change of structure $s \rightarrow s'$,

$tr : Id_S(s,s') \rightarrow (Bond_s \rightarrow Bond_{\{s'\}})$

and under forgetting (lawful parting), a **remembrance** witness remains:

$Rem(p,f) := Path_Joy(Wheat \mapsto Joy , Laughter \mapsto Joy)$

We left the field while it was still four o’clock in our bones. The wheat stayed to practice shining. The fox stayed to practice joy. The diagram—two triangles, two squares, one invisible interior—folded itself into the boy’s walk, which is how the best diagrams travel.

13) Tracks and Thirst

The switchman lived where rails cross like arguments. Levers stood in polished rows, each with a red certainty at the top. Trains came and went: full of sleeping, full of yawning, full of children with noses on glass.

“Where are they going?” the boy asked.

“No one ever knows,” said the switchman. “The first asks for the next. The next asks for the next. Only the children know what they are looking for.”

We watched a line of cars pass like beads; only the last car laughed, and only because a child waved at a field and the field waved back.

(Parenthetical code)

Routes : Category -- stations as objects, trips as arrows

Places : Obj(Routes)

Goal : Places $\rightarrow$ Joy -- meaning of arrival

Travel : People $\rightarrow$ Routes -- chosen itineraries

-- Most travelers have no witness τ : Travel $\Rightarrow$ Goal $\circ$ Arrive that commutes:

-- (No α : C $\circ$ F $\Rightarrow$ G in Joy.) Children have a natural map:

Window : Routes $\rightarrow$ Joy (press-nose functor)

τ_{child} : Window $\Rightarrow$ Goal $\circ$ Arrive (filled face)

“Why do they hurry?” the boy asked.

“Because they have no time,” said the switchman.

A merchant met us at the next siding, with a suitcase of small, perfect pills. “One pill each week,” he said, “and you will never be thirsty. Fifty-three minutes saved.”

“What will we do with the fifty-three minutes?” the boy asked.

“Anything you like,” the merchant said, meaning *nothing in particular*.

(Parenthetical code)

Let Drink : Task, cost C(Drink)= t_{water} .

Pill adds a map Shrink : Time $\rightarrow$ Time with Shrink(t_{water})=0.

But the **structure demand** G is *not* just hydration; it includes a ritual that yields

Joy: $G(s) = \text{WaterRitual}(s)$ (well, walk, sharing).

Under Joy-enrichment J : Time $\rightarrow$ Joy, the merchant supplies

α_{pill} : C'(F'(s)) $\leq$ G'(s) -- “time saved”

yet **G' shrinks**: $J(G'(s)) < J(G(s))$.

Coherence fails:

$\theta_{\text{pill}} ? : J(C') \approx J(\text{cap} \circ p')$ with $\gamma' : J(G') \sim J(\text{cap}(A))$

-- but $J(G')$ breaks the old witness; the Joy face no longer fills.

The boy held a pill like a star that had forgotten how to light. “How much is a drink of water?” he asked.

“Fifty-three minutes,” said the merchant.

“And what is it worth?”

The merchant spread his hands, which were clean from not touching rivers.

(Parenthetical code)

False optimization: lower C while silently lowering G; the “gap” reported as savings is a redefinition.

Audit flag:

if $J(G_new) < J(G_old)$ and $C_new \ll C_old$

then flag θ as broken under Joy; require an explicit collar that exhibits an equivalence of meanings

else reject the refactor

“We will keep our thirst,” the boy decided, putting the pill back like a stone on a grave. “There is a well in the desert.”

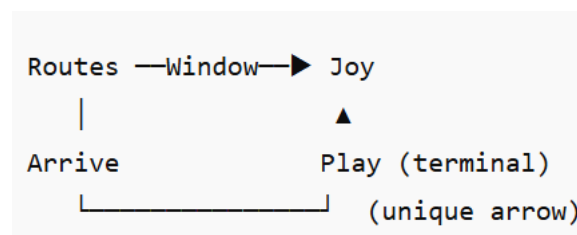
“That is far,” I said.

“Meaning is far,” he said. “But it draws.”

A freight thundered by, loaded with vegetables bound for a city that would not see their gardens. The switchman raised a hand; the train obeyed; the horizon received its weight. We walked the other way.

(Parenthetical code)

Choosing a terminal goal: children select Play : 1 -> Joy as a terminal object (corepresenting attention). For them, routes factor:



Hence they carry a canonical witness; adults seldom do.

The rails dwindled into arithmetic behind us. Sand took up its old job of being exact. Somewhere ahead was a pulley and a bucket and a sound that makes minutes into meaning.

14) The Well

We walked until the sand began to echo. At first it was only a change in the air, then a circle of stones, then a wooden frame with a wheel, then a rope, then the sound inside the sound—the song a pulley sings when it remembers water.

“There,” the boy said, though he had known for hours.

We lowered the bucket together. The rope gave back everything we had asked of it, with interest. When the bucket rose, the water was cold with the night it had kept. I poured it into the cup; the stars arrived in the surface and did not leave.

“It tastes like a gift,” the boy said.

“It tastes like waking early,” I answered, “and walking, and looking for the right dune, and finding, and being found.”

(Parenthetical code — the pushout)

We model the moment as a **gluing** of two stories along a ritual:

$$\begin{array}{ccc}
 \text{Needs} & \xleftarrow{\quad} \text{Ritual} & \xrightarrow{\quad} \text{Means} \\
 i_N & & i_M \\
 \backslash & & / \\
 \backslash & & / \\
 \perp & \text{(pushout)} & \\
 \text{Water}^* & = \text{Needs} \sqcup_{\{\text{Ritual}\}} \text{Means} &
 \end{array}$$

Universal property (gift): for any X with maps $f_N: \text{Needs} \rightarrow X$, $f_M: \text{Means} \rightarrow X$ that agree on Ritual ($f_N \circ i_N = f_M \circ i_M$), there is a unique $u: \text{Water}^* \rightarrow X$.

Here $X = \text{Joy}$; the *taste* is the mediating arrow that exists **because** walking and thirst meet in the cup.

He drank as if learning a word he had been saying all his life. I drank after him and did not apologize.

(Parenthetical code — face + interior as gluing)

Let

- $S = \{\text{thirst, fatigue, friendship-promise}\}$
- $T = \{\text{walk, draw, pour, share}\}$
- $\text{Time} = \text{minutes}$
- $R = \{\text{rope, wheel, arm, distance}\}$ with $A(s)$ from the well; $D(F(s))$ from the walk.

Faces:

$\alpha : C \circ F \leq G$ -- “today’s tasks meet the day’s demand”

$\rho : D \circ F \leq A$ -- “we have the capacity to draw”

$\text{cap} : R \rightarrow \text{Time}$ -- “capacity $\Rightarrow$ promised minutes”

The **pushout** upgrades “having both faces” to “one meaning”:

$\text{Gift} : \text{Water}^* \rightarrow \text{Joy}$ with $\text{Gift} \circ i_N = \text{NeedToJoy}$, $\text{Gift} \circ i_M = \text{MeansToJoy}$, and the ritual equality on the overlap.

Interior θ now factors through the colimit: the two time routes (α vs $\text{cap} \circ \rho + \gamma$) agree *after* gluing; taste is the witness.

We rested against the stone and watched the bucket become itself again.

“Why is it better than a faucet?” the boy asked, without scorn.

“Because a faucet forgets the proof,” I said. “It gives the answer and erases the work.”

(Parenthetical code — merchant’s pill revisited)

The pill tried to provide a map $\text{Shrink} : \text{Time} \rightarrow \text{Time}$ killing $C(\text{Drink})$, but it supplied **no** cocone on the diagram above; there was no $\text{Gift} \circ i_*$ agreeing on Ritual.

Hence no universal arrow to Joy; the refactor fails the pushout test.

He ran his palm along the rope—counting fibers, counting days. “If I had a bucket on my planet,” he said, “I would still come at four o’clock.”

“Why four?” I asked, already happy with the answer.

“Because the water is not only water at four,” he said. “It is time that remembered us.”

(Parenthetical code — dHoTT “movie” of drawing water)

Let $u \in [0,1]$ parameterize the steps: walk ($u \in [0,.4]$), draw $(.4,.7)$, pour $(.7,.9)$, share $(.9,1)$.

We get a path

Movie(u) : Process_{3D}(u)

with continuous transport of faces and a final **pushout evaluation**

$ev_1: \text{Water}^* \rightarrow \text{Joy}$.

No collar needed: rituals make the homotopy smooth.

We filled the cup a second time, because proofs are nicer when they can be reread, and because the desert had made us careful. The pulley sang again and scolded us gently for our slowness. We apologized and were forgiven.

On the stones I drew a tiny colimit symbol—a corner turned outward. The boy nodded, pleased that geometry could make a taste visible.

“When we go,” he said, “the well will stay.”

“And the witness?” I asked.

“It will stay in us,” he said. “So we can fill other cups.”

15) A Collar for Endings

We woke knowing the desert had one more exactness to teach. The boy’s eyes were steady the way a horizon is steady. “My body is heavy,” he said. “It is a shell. It will be too far to carry home.”

We walked to a place where the sand kept secrets well. A slender serpent lay there, drawn in on itself like a line deciding whether to be a circle.

“Please,” the boy said to me, “do not watch. It will look like dying. It is only a change of categories.”

He smiled, and it was a signed theorem. He kissed my cheek with the gravity of adjunctions. “I have fixed your plane for you,” he added, as if we were exchanging wrenches.

(Parenthetical code — the micro-movie / collar)

We model the passage as a **collar** around a time u^* :

$u \in [u^* - \epsilon, u^* + \epsilon]$ -- small interval

Old world: Presence New world: Memory

Maps on $[u^* - \epsilon]$: $(\eta_{\text{old}}, \epsilon_{\text{old}}, \theta_{\text{old}})$ -- taming adjunction, coherences

Maps on $[u^* + \epsilon]$: $(\eta_{\text{new}} = \emptyset, \epsilon_{\text{new}} = \text{rem})$ -- remembrance structure

Equivalence data E on the collar:

$$E : (\text{Presence}|_{\{\text{collar}\}}) \simeq (\text{Memory}|_{\{\text{collar}\}})$$

Transport: tr_E sends $\text{Bond}_{\text{old}} \mapsto \text{Remembrance}_{\text{new}}$

Endpoint evaluation:

$\text{eval}_1 : \text{Path}(u \mapsto \text{Process}(u)) \rightarrow \text{Process}(u^* + \epsilon)$ gives the *memory frame*.

He took three steps forward into privacy; the serpent wrote a single, perfect line; the sand closed like a proof completed. I kept my back to the event because he had asked me to keep the square strict—no stray arrows.

(Parenthetical code — endpoint and resource failure acknowledged)

At u^* , resource face for *staying* fails ($A_{\text{body}} < D_{\text{life}}$), but the collar supplies a lawful refactor:

Presence —Exit—► Memory

α, ρ, θ transported by E

We replace “keep rituals in body” by “keep rituals in witness,” preserving the adjunction’s snake identities in the new category.

When I turned, the desert was exactly itself. There was no body. The box with the sheep looked like a lemma from a different chapter, still true.

(Parenthetical code — persistence law)

The bond becomes

Rem(p, child) :

$\Sigma R : \text{Witness_in_Memory.}$

$(U(\varepsilon_{\text{new}}); \eta_{\text{new}} = \text{id}) \wedge (\varepsilon_{\text{new}}; F(\eta_{\text{new}}) = \text{id}) \text{ -- snake in Memory}$

A *remembrance* 2-cell R replaces proximity. Transport preserves responsibility.

I drew a tiny collar in the margin of my notebook—three frames, a bead sliding from right to left. The bead did not vanish. It became a note you could carry.

16) Body Heavy, Stars Light

Night returned like a verdict that had learned tenderness. I repaired the plane by feel and by faith; the metal made its little agreements. At dawn the engine signed its name.

I climbed until the sand withdrew from argument and became color. The sky inverted. The stars looked close enough to sort into boxes.

The laughter arrived before the clouds ended. It did not have a direction; it had a type. I tried out the new functor and found that it worked: stars became joy, not by property but by *path*.

(Parenthetical code — the laughter functor)

With the remembrance witness R from the collar, define

Laughter : Stars $\rightarrow$ Joy

Laughter := Joy $\circ$ Remember $\circ$ Bond -- induced by R

For any star σ , $\text{Laughter}(\sigma)$ is the value obtained by composing:

$\sigma \text{ --(think_of_him)--> Memory --(R)--> Joy.}$

Naturality: for constellations $f: \Sigma \rightarrow \Sigma'$,

$\text{Laughter} \circ f = \text{Laughter}$ on images (stars do not change the witness).

I spoke to no one and to one person. “If a traveler meets you,” I said into the wind, “if he sees a child who laughs and refuses a sheep that is not in a box, do not let him pass. Write to me. Quickly.”

(Parenthetical code — endpoint map of the whole story)

The story is a path

Story : Path($u \mapsto \text{Process_total}(u)$)

with endpoint evaluation

$\text{ev_1}(\text{Story}) = \text{Remembrance}$

and a derived natural transformation

Stars $\Rightarrow$ Joy (laughter).

Audit invariants for the epilogue frame:

$\text{face_coverage} = 1$ -- in Memory

$\text{theta_gap} = 0$ -- coherence by collar transport

Clouds parted to reveal a desert so exact it might have been a proof assistant. Somewhere beneath, a circle of stones kept a cup cold for strangers. I promised the stones to return them to themselves.

At a certain altitude the engine’s hum aligned with the idea that had started in a box. The world felt not larger but *connected*, the way a diagram feels when every pasted edge holds.

(Parenthetical code — univalence-ish coda)

Identity of the child is given by a *path of care*:

Child := Σ form. Path_Time(Care, Ritual) -- lived witness

Isomorphic shapes (memories, drawings) are only equivalent; the **identity** you keep is the transported witness R.

17) Audit of a Promise (coda)

Back in the world of parts and invoices, I kept a stack of blank cards in a drawer: squares for faces, nets for cubes, a little strip for collars. When children visited the hangar, I taught them to draw edges before numbers, to ask for witnesses before applause, to fill the inside only when the two routes really matched.

We made a habit of carrying water farther than necessary. We set four o'clock where we could keep it. Sometimes we drew a fox in the margin and left space for a friend.

When a grown-up asked for proof that any of it mattered, we opened the ledger we had made on the way back—columns filled not with stars but with faces that held.

(Parenthetical code — the final checklist)

V1 Edge consistency: satisfied (same maps on shared edges)

V2 Face completeness: Time face α , Resource face ρ present whenever claimed

V3 Lift check: diagonal F always factors through P when witnesses exist

V4 3D coherence: θ built from α , ρ , cap , γ ($\text{theta_gap} \rightarrow 0$)

V5 Monotonicity: cap monotone; collars preserve laws

V6 Pasting integrity: all pastes aligned; no relabel drift

V7 Movie continuity: continuous drifts or collars declared; no silent jumps

Metrics (last frame, Memory):

$\text{face_coverage}=1$, $\text{slack_time}=0$, $\text{theta_gap}=0$.

At night, when hangar lights are more honest than chandeliers, I still walk out alone and turn my face upward. The stars behave as they always did, but now they carry a small, exact laughter that is mine to keep and yours to borrow.

If you meet a child in a desert of any size who asks you to draw a sheep, give him a box. Then draw a square. Then keep your witnesses.

References

- [1] Mac Lane, S. *Categories for the Working Mathematician* (2nd ed.). Springer, 1998.
- [2] Mac Lane, S. “Natural associativity and commutativity.” *Rice University Studies* 49 (1963), 28–46.
- [3] Stasheff, J. “Homotopy associativity of H-spaces I, II.” *Transactions of the American Mathematical Society* 108 (1963), 275–312.
- [4] Joyal, A.; Street, R. “The geometry of tensor calculus I.” *Advances in Mathematics* 88 (1991), 55–112.
- [5] Kelly, G. M.; Laplaza, M. L. “Coherence for compact closed categories.” *Journal of Pure and Applied Algebra* 19 (1980), 193–213.
- [6] Selinger, P. “A survey of graphical languages for monoidal categories.” In *New Structures for Physics* (Lecture Notes in Physics 813), Springer, 2011, 289–355.
- [7] Coecke, B.; Kissinger, A. *Picturing Quantum Processes*. Cambridge University Press, 2017.
- [8] Leinster, T. *Higher Operads, Higher Categories*. Cambridge University Press, 2004.
- [9] Baez, J.; Stay, M. “Physics, topology, logic and computation: a Rosetta Stone.” In *New Structures for Physics* (LNP 813), Springer, 2011, 95–172.
- [10] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study, 2013.
- [11] Bezem, M.; Coquand, T.; Huber, S. “A model of type theory in cubical sets.” In *TYPES 2013* (post-proceedings), 2014.
- [12] Cohen, C.; Coquand, T.; Huber, S.; Mörtberg, A. “Cubical Type Theory: A constructive interpretation of the univalence axiom.” *Logical Methods in Computer Science* 14:4 (2018).
- [13] Lurie, J. *Higher Topos Theory*. Princeton University Press, 2009.

- [14] Mac Lane, S.; Moerdijk, I. *Sheaves in Geometry and Logic*. Springer, 1992.
- [15] Johnstone, P. T. *Sketches of an Elephant: A Topos Theory Compendium*. Oxford University Press, 2002.
- [16] Lawvere, F. W. “Metric spaces, generalized logic, and closed categories.” *Rendiconti del Seminario Matematico e Fisico di Milano* 43 (1973), 135–166 (1974).
- [17] Spivak, D. I. *Category Theory for the Sciences*. MIT Press, 2014.
- [18] Fong, B.; Spivak, D. I. *Seven Sketches in Compositionality: An Invitation to Applied Category Theory*. Cambridge University Press, 2019.
- [19] Loday, J.-L.; Vallette, B. *Algebraic Operads*. Springer, 2012.
- [20] Lack, S. “A 2-categories companion.” In *Towards Higher Categories*, Springer, 2010, 105–191.
- [21] Street, R. “Fibrations in bicategories.” *Cahiers de Topologie et Géométrie Différentielle* 21 (1980), 111–160.
- [22] Dixon, L.; Kissinger, A.; Merry, R. “Quantomatic: A proof assistant for diagrammatic reasoning.” *UITP 2010: User Interfaces for Theorem Provers* (workshop), 2010.
- [23] Bar, K.; Heunen, C.; Kissinger, A.; Vicary, J. “Globular: an online proof assistant for higher-dimensional rewriting.” *arXiv:1601.08105* (2016).
- [24] Homotopy.io team. “homotopy.io: interactive proofs in higher categories.” Project note, 2016–.
- [25] Youvan, D. C. *Process Geometry for CT/HoTT: Triangles-in-Squares, Tetrahedra-in-Cubes, and dHoTT Movies*. ResearchGate, September 2025. DOI: 10.13140/RG.2.2.23895.33445.
- [26] Selinger, P. “Functoriality of graphical calculi and coherence.” *Electronic Notes in Theoretical Computer Science* 143 (2006), 99–122.

[27] Carboni, A.; Kelly, G. M.; Walters, R. F. C.; Wood, R. J. “A 2-categorical approach to change of base and geometric morphisms.” *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 32 (1991), 47–95.

[28] Stasheff, J. “The associahedron and the Mac Lane–Stasheff polytopes.” In *Handbook of Algebraic Topology*, Elsevier, 1995, 275–292.

[29] Saint-Exupéry, A. de. *Le Petit Prince*. New York: Reynal & Hitchcock, 1943. (Original French edition; first publication.)

[30] Saint-Exupéry, A. de. *The Little Prince*. Translated by Katherine Woods. New York: Reynal & Hitchcock, 1943. (First English translation.)

