Pauli–Noether, Then and Now: Hydrogen's Hidden Symmetry as Algebra in Motion

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Begin where the old textbooks begin and then step aside. The hydrogen Hamiltonian reads $H = p^2/(2m) - k/r$, and angular momentum $L = r \times p$ is conserved, $[H, L_i] = 0$. Less familiar, yet equally constant, is the Runge–Lenz vector $A = (p \times L - L \times p)/2 - m k r/r$, with $[H, A_i] = 0$. In 1926 Pauli did not solve a differential equation; he listened for closure. He wrote the commutators $[L_i, L_j] = i \hbar \epsilon_{ijk} L_k$, $[L_i, A_j] = i \hbar \epsilon_{ijk} A_k$, $[A_i, A_j] = -2 m H i \hbar \epsilon_{ijk} L_k$, and on each energy shell $E<0$ he scaled $\kappa = \sqrt{-2 m E}$, $M = A/(\hbar \kappa)$, then completed the square: $J_+ = (L + M)/2$, $J_- = (L - M)/2$, so that two commuting su(2) algebras appeared and the spectrum $E_n = -m k^2/(2 \hbar^2 n^2)$ fell out as representation theory. Years earlier, Noether had taught us why this works: invariance => conservation; conserved quantities close an algebra; a closed algebra, once represented, quantizes. Read with modern eyes, the same melody becomes $so(4) \cong su(2) \oplus su(2)$, Casimirs, highest weights, and a central relation in $U(g)$. What follows is the same story told in flowing words and equations, first as Pauli and Noether could have seen it, then as abstract algebra now phrases it, with geometry (S^3, Hopf) keeping time.

Keywords: Pauli ladders, Noether invariants, Runge–Lenz vector, SO(4) symmetry, $su(2) \oplus su(2)$, hydrogen spectrum, Casimir operators, universal enveloping algebra, Hopf fibration, selection rules.

Note: 50 years ago, I first saw this 'raising and lowering operator' solution to the Hydrogen atom on the blackboard of a physical chemistry class at UC Berkeley by Professor Bruce Mahan.

Outline

1) Matrix mechanics heard as grammar.

Hydrogen as a finite alphabet with rules: $H = p^2/(2m) - k/r$; $L = r \times p$; A as above. Compute the brackets; ask only whether the symbols close. If yes, the spectrum is a theorem of the grammar, not of the coordinates.

2) Constants of motion and their algebra.

$[H, L_i] = 0$, $[H, A_i] = 0$. Rotations act by $[L_i, L_j] = i \hbar \epsilon_{ijk} L_k$ and carry A as a vector: $[L_i, A_j] = i \hbar \epsilon_{ijk} A_k$. The last line ties dynamics to symmetry: $[A_i, A_j] = -2 m H i \hbar \epsilon_{ijk} L_k$.

3) Rescaling to unit curvature and splitting in two.

Fix $E<0$, set $\kappa = \sqrt{-2 m E}$, define $M = A/(\hbar \kappa)$. Then
$[M_i, M_j] = i \epsilon_{ijk} L_k$, $[L_i, M_j] = i \epsilon_{ijk} M_k$.
Now complete the square: $J_+ = (L + M)/2$, $J_- = (L - M)/2$, and obtain
$[J_+^i, J_+^j] = i \epsilon^{ijk} J_+^k$, $[J_-^i, J_-^j] = i \epsilon^{ijk} J_-^k$, $[J_+^i, J_-^j] = 0$.

4) Casimirs crystallize energy and degeneracy.

A single identity carries the potential: $A^2 = m^2 k^2 + 2 m H (L^2 + \hbar^2)$.
With $M = A/(\hbar \kappa)$ and $\kappa^2 = -2 m E$, this yields
$J_+^2 = J_-^2 = j(j+1)$ with $j = (n-1)/2$, hence $\dim = (2j+1)^2 = n^2$ and
$E_n = - m k^2/(2 \hbar^2 n^2)$.

5) What Noether would have said.

Continuous symmetry of the action => conserved current; in quantum form, a generator commuting with H. Rotations give L; the Keplerian periapsis symmetry gives A. "The invariant operator system of hydrogen" is her phrase for Pauli's closure.

6) The abstract algebra now visible.

Let $g = \mathrm{span}\{L_i, A_i\}$ with the displayed brackets. On $E<0$, $g_E \cong so(4)$. The physics is the ideal in $U(g)$ generated by
$A^2 - m^2 k^2 - 2 m H (L^2 + \hbar^2) = 0$.
Irreps of $so(4)$ are labeled by (j_+, j_-); hydrogen selects $j_+ = j_-$.

7) Ladders as roots; selection rules as intertwiners.

Within each $su(2)$, E_+, E_- raise and lower weights; matrix elements vanish unless weights match: selection rules are $\mathrm{Hom}_g(V,W)$ statements. The old "allowed/forbidden" becomes "nonzero intertwiner."

8) Geometry keeping time: S^3 and Hopf.
On bound energy E, momentum space compacts to S^3 of radius kappa; SO(4) acts by rotation. The Hopf map $S^3 \to S^2$ exhibits two commuting rotations; equality $J_+^2 = J_-^2$ says fiber radius = base radius.

9) Beyond the textbook: Dirac, conformal echoes, and breakings.
Relativistic hydrogen keeps an enlarged symmetry (SO(4,2) whispers in the background); external fields and QED corrections break ideals and split weights. The algebra bends; the spectrum perturbs.

10) Two-proofs pedagogy and the moral.
Do it twice: (a) algebra first—closure, Casimir, highest weight => E_n; (b) analysis second—radial equation and confluent hypergeometric functions. Agreement is the moral: invariance quantizes, analysis confirms.

Epilogue (one breath).
Pauli replaced orbits with operators and solved by listening for closure; Noether had already tuned the ear. Today we simply name the chords: so(4), su(2) $\oplus$ su(2), Casimir, U(g)/I. The music is unchanged; only the notation has improved.

1) Matrix mechanics heard as grammar

Start with a tiny alphabet of symbols and the rulebook that lets them speak. The symbols are
$H = p^2/(2m) - k/r, \quad L = r \times p, \quad A = (p \times L - L \times p)/2 - m\, k\, r/r.$
The rulebook is Heisenberg's: the canonical brackets
$[r_i, r_j] = 0, \quad [p_i, p_j] = 0, \quad [r_i, p_j] = i\, \hbar\, \delta_{ij}.$
Everything that follows is syntax.

First, rotations. Compute once and reuse forever:

- $[L_i, r_j] = i\, \hbar\, \varepsilon_{ijk}\, r_k, \quad [L_i, p_j] = i\, \hbar\, \varepsilon_{ijk}\, p_k.$
  So L generates infinitesimal rotations of both r and p. Because H depends only on r and p through rotational scalars p^2 and r, we get
- $[H, L_i] = 0.$
  This already says: energy is rotationally invariant.

Next, listen for the extra consonant in the Coulomb language: A. By construction A is linear in p and affine in r, so the same rotation identities give

- $[L_i, A_j] = i\, \hbar\, \varepsilon_{ijk}\, A_k,$
  i.e., A is a genuine vector under rotations. The nontrivial line is the "self-commutator" of A. Use the product rule and the canonical brackets (a few pages of patient algebra, or a clever cyclic rearrangement) to find the dynamical clause:
- $[A_i, A_j] = -2\, m\, H\, (i\, \hbar)\, \varepsilon_{ijk}\, L_k.$
  Here the Hamiltonian appears as a *structure constant*. The language of hydrogen is not merely SO(3); H itself modulates how A curls around L.

At this point the alphabet {L, A} with the three lines

- $[L_i, L_j] = i\, \hbar\, \varepsilon_{ijk}\, L_k,$
- $[L_i, A_j] = i\, \hbar\, \varepsilon_{ijk}\, A_k,$
- $[A_i, A_j] = -2\, m\, H\, (i\, \hbar)\, \varepsilon_{ijk}\, L_k$
  is a closed grammar **if** we fix the energy shell. On an eigenspace $H|\psi\rangle = E|\psi\rangle$, replace H by the number E. For bound states $E < 0$, set $\kappa = \sqrt{-2\, m\, E}$ and rescale the unfamiliar letter to unit curvature,
- $M = A/(\hbar\, \kappa).$
  The grammar simplifies to
- $[L_i, L_j] = i\, \varepsilon_{ijk}\, L_k,$

- $[L_i, M_j] = i\, \varepsilon_{ijk} M_k$,
- $[M_i, M_j] = i\, \varepsilon_{ijk} L_k$,
  with hbar absorbed into the rescaling. This is now pure syntax: no coordinates, no differential equations—just letters and how they combine.

Complete the square in this alphabet to reveal two familiar sublanguages:

- $J_+ = (L + M)/2, \quad J_- = (L - M)/2$,
  which satisfy
- $[J_+^i, J_+^j] = i\, \varepsilon^{ijk} J_+^k, \quad [J_-^i, J_-^j] = i\, \varepsilon^{ijk} J_-^k, \quad [J_+^i, J_-^j] = 0$.
  Two commuting copies of su(2) have been hiding in the hydrogen lexicon. From here, the **theorem** about the spectrum is a theorem of grammar: a representation of su(2) ⊕ su(2) is labeled by spins (j_+, j_-), dimensions multiply, and one central identity—carrying the only trace of dynamics—ties those spins to the energy:

$$A^2 = m^2 k^2 + 2 m H (L^2 + \hbar^2).$$

Insert $M = A/(\hbar \kappa)$ and $\kappa^2 = -2 m E$, re-express in $J_\pm$, and the closure forces $J_+^2 = J_-^2 = j(j+1)$ with $j = (n - 1)/2$, hence degeneracy n^2 and

$$E_n = - m k^2 / (2 \hbar^2 n^2).$$

Nothing geometric was assumed, no radial equation was solved. We merely checked that the symbols close under the given rules. When the alphabet is finite and the grammar closes, **meaning** (the spectrum) becomes a corollary of **syntax**.

2) Constants of motion and their algebra

A constant of motion is a word that never changes its meaning as time flows. In quantum mechanics that means it **commutes with the Hamiltonian**. For hydrogen,

$$H = p^2/(2m) - k/r, \qquad L = r \times p, \qquad A = (p \times L - L \times p)/2 - m k\, r/r,$$

and the rulebook is the canonical grammar

$$[r_i, r_j] = 0, \quad [p_i, p_j] = 0, \quad [r_i, p_j] = i \hbar\, \delta_{ij}.$$

Rotational invariance. Because H depends only on the scalars p^2 and $r = |r|$, infinitesimal rotations generated by L leave H unchanged, hence

$$[H, L_i] = 0.$$

This is Noether's first line: a continuous symmetry (rotations) yields a conserved generator (angular momentum). That generator itself forms a closed sub-grammar,

$[L_i, L_j] = i \hbar \varepsilon_{ijk} L_k$,

the familiar so(3).

The extra consonant. The Coulomb problem carries more music than rotations: the Runge–Lenz vector A. It is built to transform as a vector,

$[L_i, A_j] = i \hbar \varepsilon_{ijk} A_k$,

so rotations carry A just as they carry r or p. Its constancy is the nontrivial claim:

$[H, A_i] = 0$.

Proving this is patient algebra with the product rule and the canonical brackets: expand $[H, p \times L]$, notice cancellations between kinetic and potential pieces, and watch the $-k/r$ term produce the $-mk\, r/r$ piece in A. Classically one would write $\{H, A\} = 0$ with Poisson brackets; the quantum statement is the same melody with ħ in the score.

Dynamics inside the structure constants. Two vectors that are both conserved need not commute with each other. Here they **don't**; instead their commutator curls back into L with a coefficient set by the energy:

$[A_i, A_j] = -2\, m\, H\, (i \hbar)\, \varepsilon_{ijk} L_k$.

This is the decisive line. Symmetry (A being a vector) and dynamics (H) meet in the structure constants. On an energy eigenspace $H|\psi\rangle = E|\psi\rangle$ the right-hand side becomes a number times L_k. For bound motion $E < 0$ the coefficient is positive when written with $\kappa = \sqrt{(-2\, m\, E)}$:

$[A_i, A_j] = (i \hbar)^2 \kappa^2 \varepsilon_{ijk} L_k$.

Now two further identities tighten the net:

1. **Orthogonality of the invariants.**
   $L \cdot A = A \cdot L = 0$.
   Geometrically: periapsis (A) lies in the orbital plane and points along the major axis, while L is perpendicular to that plane. Algebraically: contract $[L_i, A_j]$ with ε_{ijk} and use the Jacobi identity.

2. **The central quadratic relation.**
   $A^2 = m^2 k^2 + 2\, m\, H\, (L^2 + \hbar^2)$.
   This is the only place where m and k remember the potential. It is central because the

right-hand side commutes with L and A; it will become the "equation of state" that fixes the representation.

Collect the three commutators

$[H, L_i] = 0, \quad [H, A_i] = 0,$
$[L_i, L_j] = i \hbar \varepsilon_{ijk} L_k, \quad [L_i, A_j] = i \hbar \varepsilon_{ijk} A_k,$
$[A_i, A_j] = -2 m H (i \hbar) \varepsilon_{ijk} L_k,$

and place them next to the two scalar identities $L \cdot A = 0$ and $A^2 = m^2 k^2 + 2 m H (L^2 + \hbar^2)$. We have, at fixed energy, a **closed algebra of constants of motion**: the generators of rotations and the periapsis point, tied together by H in the structure constants and by a single quadratic relation in the center. Nothing depends on coordinates; everything depends on **commutators**. From here the path is clear: rescale A to $M = A/(\hbar \kappa)$ so the structure constants are pure numbers, split the algebra with $J_\pm = (L \pm M)/2$, and let representation theory turn constancy into quantization.

3) Rescaling to unit curvature and splitting in two

Work on a single bound-energy shell, $H|\psi\rangle = E|\psi\rangle$ with $E < 0$. The coefficient that kept slipping into the structure constants was $-2 m E$. Name its square root

$\kappa = \sqrt{-2 m E},$

so that the troublesome factor becomes κ^2. Divide the unfamiliar vector by $\hbar \kappa$ to make its curls dimensionless:

$M = A / (\hbar \kappa).$

Then the three commutators tidy themselves into pure numbers,

$[L_i, L_j] = i \varepsilon_{ijk} L_k,$
$[L_i, M_j] = i \varepsilon_{ijk} M_k,$
$[M_i, M_j] = i \varepsilon_{ijk} L_k.$

No coordinates, no parameters—just a closed algebra with **unit curvature**. The phrase means exactly this: the only scale that entered the curls has been absorbed, so the remaining structure constants are ±1. (Had we chosen $E > 0$, the sign would flip and the algebra would read like so(3,1); the bound spectrum is the so(4) branch.)

Now complete the square in operator form. Add and subtract the two vector letters to decouple them:

$J_+ = (L + M)/2, \quad J_- = (L - M)/2.$

Invert immediately to remember what we did:

$L = J_+ + J_-, \quad M = J_+ - J_-.$

One Jacobi-identity's worth of algebra (or, more playfully, expand and regroup with the three lines above) yields the commutators

$[J_+^i, J_+^j] = i\, \varepsilon^{ijk} J_+^k,$
$[J_-^i, J_-^j] = i\, \varepsilon^{ijk} J_-^k,$
$[J_+^i, J_-^j] = 0.$

Two independent su(2)'s fall out of the bracket, commuting past each other like twin gears that never touch. The geometry hiding behind the rescaling becomes visible: $so(4) \cong su(2) \oplus su(2)$ is the Lie algebra of rotations of a 3-sphere S^3 of **unit radius**. Our κ rescaling was exactly the move that normalized the sphere in momentum space to radius 1; the operators $J_\pm$ are the infinitesimal generators of the two commuting left/right rotations of S^3.

Three quick checks keep the picture honest:

1. Orthogonality survives the change of letters. From $L \cdot M = 0$ we get $J_+^2 - J_-^2 = 0$, so the two spins are constrained to have equal Casimirs on a bound shell.
2. The old algebra is recovered by recombining: insert $L = J_+ + J_-$ and $M = J_+ - J_-$ into the $su(2) \oplus su(2)$ brackets to re-derive [L,L], [L,M], [M,M] with unit coefficients.
3. Curvature bookkeeping: if $E < 0$ we rescale by κ and land on so(4); if $E > 0$ we would rescale by $\sqrt{2 m E}$ and obtain $[M,M] = -\, i\, \varepsilon\, L$, i.e., so(3,1). The splitting is therefore the algebraic fingerprint of closed (elliptic) vs open (hyperbolic) Kepler motion.

After the split, the hydrogen alphabet has become two familiar dialects spoken at once. Each $J_\pm$ carries its own ladder operators $J_\pm^\pm$ that raise and lower weights within su(2); because $[J_+, J_-] = 0$ the ladders interleave without interference. What remains, in the next movement, is to relate the sizes of these two spins to the dynamics. That tie is the single quadratic sentence that still remembers m and k:

$A^2 = m^2 k^2 + 2 m H (L^2 + \hbar^2),$

which, rewritten in the $J_\pm$ letters and with $\kappa^2 = -2 m E$, will force $J_+^2 = J_-^2$ and quantize their common value—turning unit curvature into discrete energy.

4) Casimirs crystallize energy and degeneracy

All the dynamics hide in one quadratic sentence:

$A^2 = m^2 k^2 + 2 m H (L^2 + \hbar^2)$.

Pass to an energy shell $H|\psi\rangle = E|\psi\rangle$ with $E < 0$ and name the bound-state curvature

$\kappa^2 = -2 m E$ (> 0).

Divide the central relation by $\hbar^2 \kappa^2$ and use $M = A/(\hbar \kappa)$. Remember that in the algebra of §3 we have already measured L in units of hbar (so $[L_i,L_j] = i \epsilon_{ijk} L_k$). The sentence becomes

$M^2 = (m^2 k^2)/(\hbar^2 \kappa^2) + (2 m E / \kappa^2)(L^2 + 1)$
$= (m^2 k^2)/(\hbar^2 \kappa^2) - (L^2 + 1)$,

hence

$L^2 + M^2 = (m^2 k^2)/(\hbar^2 \kappa^2) - 1$. (★)

Now complete the square one last time. With

$J_+ = (L + M)/2$, $J_- = (L - M)/2$,

and $L \cdot M = 0$, the su(2) Casimirs are

$J_+^2 = (1/4)(L^2 + M^2)$, $J_-^2 = (1/4)(L^2 + M^2)$.

So the two spins coincide:

$J_+^2 = J_-^2 = (1/4)[(m^2 k^2)/(\hbar^2 \kappa^2) - 1]$. (◆)

But in su(2) every irreducible says its own name: $J_\pm^2 = j_\pm(j_\pm + 1)$. The equality above forces

$j_+ = j_- = j$, $j(j+1) = (1/4)[(m^2 k^2)/(\hbar^2 \kappa^2) - 1]$.

Solve the bracket:

$(m^2 k^2)/(\hbar^2 \kappa^2) = 4 j(j+1) + 1 = (2 j + 1)^2$,

so the bound-state curvature is quantized,

$\kappa = (m k)/(\hbar (2 j + 1))$,

and therefore the energy is

$E = - \kappa^2/(2 m) = - m k^2 / [2 \hbar^2 (2 j + 1)^2]$.

Name the principal quantum number $n = 2 j + 1$ ($n = 1,2,3,\ldots$). Then

$E_n = - m k^2 / (2 \hbar^2 n^2)$,

and the degeneracy is the size of the irrep of $su(2) \oplus su(2)$:

$\dim = (2 j_+ + 1)(2 j_- + 1) = (2 j + 1)^2 = n^2$.

The spectrum is now a corollary of the Casimir, the degeneracy a product of two equal spins. One central identity (★), one completion of the square (◆), and hydrogen's numbers fall out as grammar—curvature quantized, energy discrete, multiplicity squared.

5) What Noether would have said

Begin from the action, not the equation of motion:

$S[r] = \int (\frac{1}{2} m |\dot{x}|^2 + k/|r|) dt$.

A continuous variation $r \to r + \delta r$ that changes S only by a total time derivative produces a conserved quantity. This is Noether's sentence; everything else is grammar.

The familiar symmetry is rotation, $\delta r = \varepsilon \times r$ with constant ε. The Noether charge is

$L = r \times p, \quad p = m \dot{x}$,

and indeed $dL/dt = 0$ because S is invariant under SO(3). In Poisson language,

$\{L_i, L_j\} = \varepsilon_{ijk} L_k, \quad \{H, L_i\} = 0, \quad H = p^2/(2m) - k/r$.

But the Kepler action carries a subtler invariance, a *dynamical* one that shifts the periapsis while preserving energy. Its generator is linear in p and affine in r, and Noether's current is

$A = p \times L - m k\, r/r$,

the Laplace–Runge–Lenz vector. One can characterize the associated variation implicitly: let $G(n) = n \cdot A$ for a fixed direction n; then the phase–space flow $(\delta r, \delta p)$ it generates through Hamilton's rule

$\delta r = \eta \{r, G(n)\}, \quad \delta p = \eta \{p, G(n)\}$,

is a symmetry of the action ($\delta\eta$ S is a total derivative), whence $d(n \cdot A)/dt = 0$. Thus A is not an accident of the equations of motion; it is a Noether charge of a continuous Coulomb-specific transformation that leaves the action invariant up to boundary terms. In the compact language she preferred: *a further one–parameter group of the variational problem exists besides rotations; its invariant current is A.*

Noether's viewpoint arranges the algebra automatically. Every symmetry gives a generator; the generators inherit a Lie algebra from the composition of the underlying transformations. Rotations act on vectors,

$$\{L_i, r_j\} = \varepsilon_{ijk} r_k, \quad \{L_i, p_j\} = \varepsilon_{ijk} p_k, \quad \{L_i, A_j\} = \varepsilon_{ijk} A_k,$$

so A is a vector current. Two such currents need not commute; their bracket measures how the corresponding transformations compose. For hydrogen,

$$\{A_i, A_j\} = -2\, m\, H\, \varepsilon_{ijk} L_k,$$

so dynamics (H) appear as a structure constant. Quantization replaces Poisson brackets with $(1/i\hbar)$ times commutators, turning Noether currents into selfadjoint operators and the algebra into

$$[L_i, L_j] = i\, \hbar\, \varepsilon_{ijk} L_k, \quad [L_i, A_j] = i\, \hbar\, \varepsilon_{ijk} A_k, \quad [A_i, A_j] = -2\, m\, H\, (i\, \hbar)\, \varepsilon_{ijk} L_k.$$

This is exactly Pauli's closure, now seen as the operator algebra of Noether charges.

Two scalar facts are already encoded in the variational geometry. First, orthogonality:

$$L \cdot A = 0,$$

for the extra transformation shifts the *direction* of the periapsis inside the orbital plane without tilting the plane itself; the plane normal L is untouched. Second, a central quadratic identity that remembers the potential:

$$A^2 = m^2 k^2 + 2\, m\, H\, (L^2 + \hbar^2).$$

Classically the $\hbar^2$ term is absent; in quantum mechanics it appears from operator ordering and ensures that the right–hand side commutes with all generators—Noether would call it an invariant of the full symmetry.

At fixed energy E, the algebra of these currents rescaled to unit curvature becomes

$$[L_i, L_j] = i\, \varepsilon_{ijk} L_k, \quad [L_i, M_j] = i\, \varepsilon_{ijk} M_k, \quad [M_i, M_j] = i\, \varepsilon_{ijk} L_k,$$

with $M = A/(\hbar\, \kappa)$, $\kappa^2 = -2\, m\, E$. The group integrated by this Lie algebra is SO(4), the rotation group of a 3–sphere; its two commuting SU(2) factors appear by the elementary change of letters

$$J_\pm = (L \pm M)/2.$$

In Noether's idiom one would not say "$so(4) \cong su(2) \oplus su(2)$," but rather: *the invariant operator system of the hydrogen variational problem decomposes into two independent systems of* *infinitesimal rotations*, and the energy shell selects representations with equal quadratic invariants,

$J_+^2 = J_-^2$.

From there the quantization is representation theory in a single line:

$J_\pm^2 = j(j+1) \Rightarrow (2j+1) = n \Rightarrow E_n = - m k^2/(2 \hbar^2 n^2)$,

and the multiplicity is the product of dimensions,

$\dim = (2j+1)^2 = n^2$.

So Pauli's "matrix trick" is Noether's theorem written with operators. A symmetry of the action begets a conserved current; the currents close; a central identity trims the possibilities; a representation fixes the spectrum. The periapsis slides because a one–parameter invariance says it may; its current is A; its algebra with rotations is hydrogen's hidden grammar.

6) The abstract algebra now visible

Strip away coordinates and keep only the letters and their curls. Let

$g = \mathrm{span}\{ L_i, A_i \}$

with the brackets already sung:

$[L_i, L_j] = i \hbar \varepsilon_{ijk} L_k$,
$[L_i, A_j] = i \hbar \varepsilon_{ijk} A_k$,
$[A_i, A_j] = -2 m H (i \hbar) \varepsilon_{ijk} L_k$.

Fix an energy shell $H|\psi\rangle = E|\psi\rangle$ with $E < 0$ and divide out the curvature. With $\kappa^2 = -2 m E$ and $M = A/(\hbar \kappa)$, the same relations become

$[L_i, L_j] = i \varepsilon_{ijk} L_k$,
$[L_i, M_j] = i \varepsilon_{ijk} M_k$,
$[M_i, M_j] = i \varepsilon_{ijk} L_k$.

This is a Lie algebra with unit structure constants; call it g_E. The change of letters

$J_+ = (L + M)/2, \quad J_- = (L - M)/2$

exhibits

$[J_+^i, J_+^j] = i \varepsilon^{ijk} J_+^k, \quad [J_-^i, J_-^j] = i \varepsilon^{ijk} J_-^k, \quad [J_+^i, J_-^j] = 0$, so $g_E \cong su(2) \oplus su(2)$ and, integrating, $G_E \cong SO(4)$. At this level the algebra is *pure syntax*: two commuting rotation algebras acting together.

Physics re-enters as a **central sentence**—a relation that lives in the center and trims the free words to hydrogenic speech:

$$A^2 - m^2 k^2 - 2 m H (L^2 + \hbar^2) = 0.$$

Transport it to the unit-curvature alphabet. Divide by $\hbar^2 \kappa^2$ and use $M = A/(\hbar \kappa)$, $\kappa^2 = -2 m E$:

$$M^2 + L^2 = (m^2 k^2)/(\hbar^2 \kappa^2) - 1. \quad (\flat)$$

In the universal enveloping algebra $U(g_E)$ this is an **ideal generator**: we pass from $U(g_E)$ to the quotient

$$\mathcal{A}_E = U(g_E) / \langle M^2 + L^2 - c_E \cdot 1 \rangle,$$

where $c_E = (m^2 k^2)/(\hbar^2 \kappa^2) - 1$ is a scalar on the shell. Thus “the physics” is precisely the choice of ideal that kills every word except those consistent with ($\flat$). In representation-theoretic language: hydrogen chooses a **central character**.

Irreducible *unitary* representations of $so(4) \cong su(2) \oplus su(2)$ are labeled by a pair of spins (j_+, j_-). Their quadratic Casimirs read

$$J_+^2 = j_+(j_+ + 1), \quad J_-^2 = j_-(j_- + 1),$$

and the dimension multiplies:

$$\dim = (2 j_+ + 1)(2 j_- + 1).$$

But L and M recombine as $L = J_+ + J_-$ and $M = J_+ - J_-$. Hence

$$L^2 + M^2 = 2(J_+^2 + J_-^2). \quad (\sharp)$$

Insert ($\sharp$) into ($\flat$):

$$2(J_+^2 + J_-^2) = c_E,$$

i.e.,

$$j_+(j_+ + 1) + j_-(j_- + 1) = c_E/2.$$

A second scalar constraint comes from orthogonality $L \cdot M = 0$. In the $J_\pm$ letters it is

$$J_+^2 - J_-^2 = 0,$$

forcing the spins to **match**:

$$j_+ = j_- = j.$$

Now both constraints collapse to one:

$2 \cdot j(j + 1) = c_E/2 \quad \Rightarrow \quad c_E = 4 j(j + 1)$.

Return to c_E's definition:

$(m^2 k^2)/(\hbar^2 \kappa^2) - 1 = 4 j(j + 1)$,

so

$(m k)/(\hbar \kappa) = 2 j + 1$.

Name $n = 2 j + 1 \in \{1,2,3,\ldots\}$. Then $\kappa = (m k)/(\hbar n)$ and the energy reads

$E_n = - \kappa^2/(2 m) = - m k^2/(2 \hbar^2 n^2)$,

while the degeneracy becomes the product of the twin spins:

$\dim = (2 j_+ + 1)(2 j_- + 1) = n^2$.

In a single paragraph: choose the Lie algebra $g_E = so(4)$; impose the hydrogen ideal in $U(g_E)$; the ideal fixes the **highest weights** to $j_+ = j_-$; the common weight determines κ and therefore E_n; multiplicities square because two equal spins multiply. Pauli's closure has become an algebra object ($\mathcal{A}_E$), Noether's invariants are its generators, and the spectrum is the list of irreps permitted by the central character. Geometry nods in agreement: S^3 carries two commuting rotations; equality of their radii is $J_+^2 = J_-^2$; quantization is the statement that only integer or half-integer radii are allowed.

7) Ladders as roots; selection rules as intertwiners

Two spins speak at once. Choose Cartans
$H_+ = J_+^3, \quad H_- = J_-^3$,
and their root words (ladder letters)
$E_+^{\pm} = J_+^1 \pm i J_+^2, \quad E_-^{\pm} = J_-^1 \pm i J_-^2$.
They satisfy the su(2) grammar twice over:

$$[H_\pm, E_\pm^\pm] = \pm E_\pm^\pm, [H_\pm, E_\pm^\mp] = \mp E_\pm^\mp, [E_\pm^+, E_\pm^-] = 2H_\pm, [J_+^a, J_-^b] = 0.$$

Work in the weight basis

$$| \; j_+, m_+; \; j_-, m_- \rangle, m_\pm = -j_\pm, -j_\pm + 1, \ldots, +j_\pm,$$

with

$$H_{\pm} \mid j_+, m_+; j_-, m_- \rangle = m_{\pm} \mid j_+, m_+; j_-, m_- \rangle,$$

and the usual ladders

$$E_+^+ \mid j_+, m_+; \cdot \rangle \propto \mid j_+, m_+ + 1; \cdot \rangle, E_-^+ \mid \cdot; j_-, m_- \rangle \propto \mid \cdot; j_-, m_- + 1 \rangle,$$

(and similarly for $E_{\pm}^-$ lowering). Because $[J_+, J_-] = 0$, the two ladders interleave without friction: you may raise m_+ holding m_- fixed, or vice versa.

Physical rotations are the **diagonal** su(2) with generators

$$L = J_+ + J_-, L_{\pm} = L^1 \pm iL^2,$$

so the magnetic quantum number seen in the lab is

$$m = m_+ + m_-.$$

Its selection rule is the familiar one from a single spin: $L_{\pm}$ changes m by ± 1. Inside a fixed principal shell n (hence $j_+ = j_- =: j$), all n^2 states are the weights of the (j, j) irrep; the four letters $E_{\pm}^{\pm}$ simply **navigate** that square lattice of (m_+, m_-).

Now speak the language of **intertwiners**. An operator T is a g-intertwiner $V \to W$ when it commutes with the action:

$$T \in \mathrm{Hom}_g(V, W) \Leftrightarrow T\, \rho_V(X) = \rho_W(X)\, T \quad \forall\, X \in g.$$

If T is a true scalar under g, Schur's lemma says

$$\mathrm{Hom}_g((j_+, j_-), (j_+', j_-')) = \begin{cases} \mathbb{C}, & (j_+', j_-') = (j_+, j_-) \\ \{0\}, & \text{otherwise}, \end{cases}$$

so **matrix elements vanish** unless the so(4) labels match. That is the abstract form of "forbidden."

More generally, physical observables transform as **tensors** under a chosen symmetry. Under the diagonal rotations (the lab's SO(3)), the position operator r is rank-1, and the Wigner–Eckart theorem says

$$\langle n'\ell'm' \mid r_q \mid n\ell m \rangle = \langle \ell'\, m' \mid 1\, q;\ \ell\, m \rangle\, \langle n'\ell' \parallel r \parallel n\ell \rangle,$$

so the *only* nonzero angular pieces are those with

$$\Delta\ell = \pm 1, \Delta m = 0, \pm 1,$$

and parity forbids $\Delta\ell = 0$. “Allowed/forbidden” reduces to “Clebsch–Gordan coefficient nonzero.”

Viewed at the **so(4)** level, $r$sits in a tensor type that couples $(j,j)$to $(j \pm \frac{1}{2}, j \pm \frac{1}{2})$. Because $n = 2j + 1$, this enforces

$$\Delta n = \pm 1$$

for electric-dipole transitions (the radial reduced matrix element kills $\Delta n = 0$in the Coulomb problem). Thus the textbook rules

$$\Delta n = \pm 1, \Delta\ell = \pm 1, \Delta m = 0, \pm 1$$

are nothing but **representation bookkeeping**: $r$is not an intertwiner $(j,j) \to (j,j)$, so $\mathrm{Hom}_{\mathrm{so}(4)}((j,j),(j,j))$gives zero; instead $r$is an arrow into the adjacent highest weights, and only those Hom spaces are nontrivial.

Inside a fixed n, the generators $L_i$and $M_i$themselves are **tangent** operators: they are components of the so(4) action, hence do not change $(j_+, j_-)$and so do not change n. Their ladders $E_\pm^\pm$merely reshuffle weights (m_+, m_-)within the same irrep—degeneracy navigation, not spectroscopy.

In one line: ladders are the **root vectors** of su(2) ⊕ su(2); selection rules are the statement that a matrix element is a **g-equivariant map**—a nonzero intertwiner—between the source and target irreps. When the Hom space vanishes, the transition is forbidden by algebra, not by accident.

8) Geometry keeping time: S^3 and Hopf

Bound energy means closed orbits and compact kinematics. Fock’s trick makes this literal: for E<0 let κ=√(−2mE) and send each momentum p∈ℝ^3 to a point ξ on the unit 3–sphere S^3⊂ℝ^4 by stereography,

$$\xi_0 = \frac{p^2 - \kappa^2}{p^2 + \kappa^2}, \boldsymbol{\xi} = \frac{2\kappa\,\mathbf{p}}{p^2 + \kappa^2}, \xi_0^2 + |\,\boldsymbol{\xi}\,|^2 = 1.$$

Hydrogen's bound shell is thus a **round S^3 of radius 1**; its symmetry is rotation of that sphere:

$$\mathrm{SO}(4) \text{ acts by } \xi \mapsto R\xi.$$

Our algebra already knew this: with M=A/(\hbar\kappa) the unit–curvature brackets

$$[L_i, L_j] = i\varepsilon_{ijk}L_k, [L_i, M_j] = i\varepsilon_{ijk}M_k, [M_i, M_j] = i\varepsilon_{ijk}L_k$$

identify the Lie algebra so(4). Completing the square,

$$J_+ = \frac{1}{2}(L+M), J_- = \frac{1}{2}(L-M),$$

exposes **two commuting su(2)s**, the left– and right–invariant rotations of S^3.

Now pass from coordinates to spinors. Regard S^3 as the unit sphere in ℂ^2:

$$z = (\genfrac{}{}{0pt}{}{z_1}{z_2}), \| z \|^2 = | z_1 |^2 + | z_2 |^2 = 1.$$

Left and right actions of SU(2) are

$$z \qquad \mapsto \qquad U_+ z, z$$

with infinitesimal generators precisely $J_+$and J_-. The **Hopf map** sends S^3 of spinors to the Bloch 2–sphere:

$$\mathbf{n} = z^\dagger \boldsymbol{\sigma} z \qquad \in S^2,$$

where $\boldsymbol{\sigma}$are the Pauli matrices. The fiber over a point $\mathbf{n}$is the phase circle

$$z \qquad \sim \qquad e^{i\phi}z, S^3$$
$$\longrightarrow Hopf S^2, \text{ fiber } U(1).$$

Geometrically: **two commuting rotations** live on S^3—left and right. The **diagonal** rotation $L = J_+ + J_-$descends to ordinary rotations of $\mathbf{n}$on S^2 (what the lab sees). The **difference** $J_+ - J_-$has a component along the U(1) fiber (a hidden phase motion) and a component mixing base and fiber—the algebraic shadow of periapsis motion generated by A.

The Casimirs are radii. On any irrep (j_+, j_-),

$$J_+^2 = j_+(j_+ + 1), J_-^2 = j_-(j_- + 1).$$

Hydrogen's central relation forces **equality**,

$$J_+^2 = J_-^2 \Leftrightarrow j_+ = j_-,$$

which says: the left and right rotations carry **equal metric weight**. On the round S^3 this reads as **fiber radius = base radius** in the bi–invariant metric; the Hopf fibration is balanced. Representation–theoretically, the bound level $n$is the irrep $(j, j)$with $n = 2j + 1$, hence multiplicity

$$\dim (j, j) = (2j + 1)^2 = n^2,$$

the number of hyperspherical harmonics of degree $n - 1$on S^3. In momentum language, wavefunctions are just spherical harmonics **on the 3–sphere**, rotated by SO(4); in spinor language, they are polynomials in $z$that transform with the **same** spin on the left and on the right; in the lab, they decompose into familiar $Y_{\ell m}$after the Hopf projection.

Thus the line $J_+^2 = J_-^2$is not a modest algebraic coincidence; it is the metric statement that hydrogen's bound dynamics live on a **perfectly round** S^3 where left and right wheels turn with the same radius. The Hopf map keeps time: base S^2 is what we measure, fiber U(1) is what we don't—until selection rules let a tensor reach across.

9) Beyond the textbook: Dirac, conformal echoes, and breakings

Turn up the tempo to c ≠ ∞. The Dirac Hamiltonian for a point Coulomb field is
H_D = c α·p + β m c^2 − k/r,
with {α_i, α_j} = 2 δ_ij, {α_i, β} = 0, β^2 = 1. Its spectrum preserves the Coulomb melody but modulates the chord: energies depend on the total angular momentum j (not on ℓ alone),
E_{n j} = m c^2 \Big[1 + \frac{(Zα)^2}{(n − δ_j)^2} \Big]^{−1/2},
δ_j = j + 1/2 − \sqrt{(j + 1/2)^2 − (Zα)^2}.
Here the hidden SO(4) is not destroyed but **embedded**; a conserved "Johnson–Lippmann" operator completes the Dirac algebra so that the dynamical symmetry enlarges toward the conformal cloak SO(4,2). You can hear it in the degeneracy: for fixed n, states with the same j remain paired even as ℓ-splitting appears. The old letters L and A get dressed: L becomes the generator of the diagonal SU(2) acting on spinor harmonics, while A deforms into a first-order

operator that still commutes with H_D and anticommutes with the Dirac $K = \beta(\Sigma\cdot L + \hbar)$, keeping a remnant of the hydrogenic square.

Slow back down with a Foldy–Wouthuysen glide: project H_D to the positive-energy block and expand in $(v/c)^2$. The **Breit–Pauli** Hamiltonian appears as a sequence of central relations added to the nonrelativistic grammar, each a gentle breaking of so(4):
$H_{BP} = H + H_{kin}^{(rel)} + H_{SO} + H_{Darwin} + \ldots,$
$H_{kin}^{(rel)} = - p^4/(8 m^3 c^2),$
$H_{SO} = (1/(2 m^2 c^2 r))(dk/dr) L\cdot S,$
$H_{Darwin} = (\pi \hbar^2/(2 m^2 c^2)) k \delta(r).$
These terms do not commute with the full hydrogen algebra; the ideal in U(g) thickens into **almost**-central constraints, and equal Casimirs $J_+^2 = J_-^2$ begin to **tilt** by $O(\alpha^4)$. External fields continue the bending: a weak B adds $-\mu_B (L + 2 S)\cdot B$ (normal Zeeman) or its Paschen–Back limit; an E field couples dipolarly and mixes n with n ± 1 (Stark). In each case, selection rules become intertwiners for a **smaller** symmetry—SO(3) in B, the parabolic subgroup in E—and weights split according to the surviving quantum numbers (m_j in Zeeman, parabolic indices in Stark).

Finally, the whisper becomes a correction in the ear itself: QED. Self-energy and vacuum polarization modify the central relation by logarithms and contact terms; the **Lamb shift** lifts the Dirac j-degeneracy ($2S_{1/2}$ vs $2P_{1/2}$), a spectral line drawn by the photon field inside the algebra. If SO(4) was the round S^3 with equal left/right radii, then relativity makes it **slightly oblate**, external fields **tilt the axis**, and QED **wrinkles the metric**. Yet the score is recognizable: ladders persist as local coordinates on broken or deformed orbits; intertwiners shrink to the symmetry that remains; and each new term is a small sentence added to the ideal that trims the universal envelope into physics. The algebra bends; the spectrum perturbs; the melody of invariance keeps the time.

10) Two-proofs pedagogy and the moral

Do it twice so the mind learns both the grammar and the melody.

(a) Algebra first. Begin with closure:
$[L_i,L_j] = i \hbar \epsilon_{ijk} L_k$, $[L_i,A_j] = i \hbar \epsilon_{ijk} A_k$, $[A_i,A_j] = -2 m H (i \hbar) \epsilon_{ijk} L_k$.
Fix $E<0$, set $\kappa^2 = -2 m E$, $M = A/(\hbar \kappa)$, complete the square:
$J_+ = (L+M)/2$, $J_- = (L-M)/2$, $[J_+^i,J_+^j] = i \epsilon^{ijk} J_+^k$, $[J_-^i,J_-^j] = i \epsilon^{ijk} J_-^k$.
One central sentence carries the dynamics:
$A^2 = m^2 k^2 + 2 m H (L^2 + \hbar^2).$
Divide by $\hbar^2 \kappa^2$, use M and $\kappa^2 = -2 m E$, get

L^2 + M^2 = (m^2 k^2)/(hbar^2 kappa^2) - 1.
But L^2 + M^2 = 2(J_+^2 + J_-^2) and L·M = 0 gives J_+^2 = J_-^2 = j(j+1). Solve for the curvature:
(m k)/(hbar kappa) = 2 j + 1 = n, hence E_n = - m k^2/(2 hbar^2 n^2), dim = (2j+1)^2 = n^2.
The spectrum is a theorem of syntax.

(b) Analysis second. Separate variables in coordinates and let the calculus do its work. Write the stationary Schrödinger equation
[-(hbar^2/2m) nabla^2 - k/r] psi = E psi, with E<0.
Choose spherical coordinates, psi(r,theta,phi) = R_{n l}(r) Y_{l m}(theta,phi). Angular separation yields L^2 Y_{l m} = hbar^2 l(l+1) Y_{l m}. The radial equation becomes
R'' + (2/r) R' + [2mE/hbar^2 + 2mk/(hbar^2 r) - l(l+1)/r^2] R = 0.
Introduce the bound-state scale a = hbar^2/(m k) and the dimensionless variable rho = 2 r/(n a) to anticipate hydrogenic decay. Write R(r) = u(r)/r and impose u ~ e^{-rho/2} rho^{l+1} v(rho).
The equation for v is Kummer's confluent hypergeometric form,
rho v'' + (2l+2 - rho) v' + (n - l - 1) v = 0.
Polynomial termination (regularity at infinity) requires n - l - 1 to be a nonnegative integer, fixing n = 1,2,3,..., with 0 <= l <= n-1.
Normalizability then enforces the same energy:
E_n = - m k^2/(2 hbar^2 n^2),
and the degeneracy counts the allowed Y_{l m} across l = 0..n-1:
sum_{l=0}^{n-1} (2l+1) = n^2.
The spectrum is a theorem of analysis.

Now set the two proofs side by side. Algebra heard a closed Lie algebra, imposed one central quadratic identity, and read off highest weights; analysis solved a second-order ODE by taming its asymptotics and forcing polynomials. They agree digit for digit: energies, multiplicities, and selection rules align—J_+^2 = J_-^2 on S^3 matches n = 2j+1 from the Kummer index; dim(j,j) = n^2 matches the spherical count.

Epilogue (one breath)

Pauli set the telescope down and picked up a tuning fork: orbits became operators, ellipses became brackets, and the spectrum arrived by listening for closure; Noether had already tuned the ear, proving that a continuous sway of the action begets a steady note in time, so that L and A—rotations and periapsis—sing in counterpoint until the harmony locks as [L,L]∼L, [L,A]∼A, [A,A]∼H·L, and a single quadratic refrain, $A^2 = m^2k^2 + 2mH(L^2 + \hbar^2)$, fixes the key; today we merely name the chords we hear—so(4) for the stage, su(2) ⊕ su(2) for the twin hands, Casimir for the tempo that won't change, and U(g)/I for the score with one decisive bar line—while geometry keeps time on S^3 and the Hopf drum beats base and fiber in equal radius, $J_+^2 = J_-^2$, $n = 2j + 1$, $E_n = -mk^2/(2\hbar^2 n^2)$; the melody is unchanged—symmetry quantizes, analysis confirms—only the notation has improved, the old music now printed cleanly, sharps and flats labeled, yet still the same hydrogen song you can whistle without coordinates.

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(Where section numbers in our text point: §§1–4 lean on Pauli/Fock/Bargmann; §5 on Noether; §6 on U(g)/I and representation theory à la Bander–Itzykson; §7 on Wigner–Eckart; §8 on Hopf/Fock; §9 on Dirac, Johnson–Lippmann, FW, Breit–Pauli, and QED; §10 ties back to Schrödinger's analytic solution.)